Is Subsidizing Inefficient Bidders Actually Costly?

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A widespread practice, particularly in public-sector procurement and dispersal, is to subsidize a class of competitors believed to be at an economic disadvantage. Arguments for such policies vary, but they typically assume that benefits of subsidization must be large enough to outweigh a presumed economic cost of the subsidy. When disadvantaged competitors compete in auctions, the subsidy serves to make them more competitive rivals. Other bidders rationally respond by bidding more aggressively. We consider a model of procurement auctions and show that a policy of subsidizing inefficient competitors can lower expected project cost and also enhance economic efficiency. Some subsidy is generally better than no subsidy for a wide range of parameters.
(*Auctions; Subsidies; Disadvantaged Competitors; Affirmative Action; Set-Asides; Procurement Costs*)

1. Introduction

From time to time, certain classes of competitors are given explicit advantages in competitions, such as auctions. Favored classes have included veterans, racial minorities, women, and small businesses.[1] Often, the classes accorded advantages are considered, at least on average, to be disadvantaged (i.e., less effective competitors). The rationale for the advantage frequently stems from important noneconomic aspects—past service or discrimination, populism in general, or notions of fairness. Sometimes, a class of competitors is favored via quotas or set-asides. However, often the advantage takes the form of subsidies, discounts, credits, or special payment terms should a disadvantaged competitor win.[2] Such special payment or compensation terms are analyzed in this paper in an auction setting. While the rationale for such special treatment is often based upon hotly debated considerations of fairness, it is widely presumed that such favoritism is costly for the bidtaker and economically inefficient.[3] We show that neither presumption is necessarily correct.[4]

[1] Veterans have been given explicit advantages on civil-service examinations. In some of the U.S. Federal Communications Commission's auctions of radio spectrum rights, preferred status has been given to firms owned by minorities or women, "small" businesses, and rural telephone companies.

[2] In the spectrum auctions, favored bidders were given both direct price discounts and partial financing of the bulk of payments at favorable interest rates (Federal Communications Commission 1996).

[3] "Civil rights advocates have implicitly conceded that affirmative action subsidies burden the public fisc—they argue instead that the social benefits of remedying past discrimination or of promoting diversity justify the cost of the government subsidies." (Ayres and Cramton 1996, p. 450).

[4] This paper does take seriously the notion that the bidders designated to receive bidding subsidies are, in fact, disadvantaged. If bidding credits were to be granted merely on the basis of political

0025-1909/03/4901/0071$5.00
1526-5501 electronic ISSN

Consider sealed bidding for a construction project. Suppose that there are several bidders and that, while the cost of the job is uncertain, it is clear to all concerned that it will cost one of the bidders (the "disadvantaged" bidder)[5] about 20% more than it would the others to do the job. Should the bidtaker offer publicly to subsidize the disadvantaged bidder? For example, by offering to pay him 12% more than the amount of his bid, if he makes the lowest bid? This paper sets aside redistributive or other political reasons for subsidy policies, to focus on purely economic considerations: the bidtaker's expected project cost may well be reduced by offering such a subsidy. Furthermore, suppose the bidtaker is a government body that must raise revenue via distortive taxes. Then, the economic efficiency impact of the reduction in expected project cost may well outweigh the allocative inefficiency caused by the increased chance that the disadvantaged bidder will win the job. Thus, surprisingly, in these circumstances, overall economic efficiency *requires* a government bidtaker to subsidize the inefficient competitor.

This message might be considered as an implication, if an often unrecognized one, of the optimal auctions literature. Myerson (1981) considers bidders seeking to buy, under the independent private-values assumption: that potential bidders' valuations of the auctioned asset are independent draws from known (possibly asymmetric) distributions. He finds that expected revenue maximization implies policies that can be thought of as subsidizing bidders who are more likely to draw low valuations.

However, to use the independent private-values assumption for analysis of such real economic policy questions is to make the untenable assumption that bidders face no related (correlated) cost uncertainties, such as potential weather delays or future materials prices.[6] This paper adapts the common-value auction model[7] to the task of understanding the real costs of subsidizing disadvantaged bidders when all bidders realistically face correlated cost uncertainties.

To analyze such situations, it is necessary to deal with complicated asymmetric auctions. In general, it has proven difficult to find Nash equilibria when asymmetric bidders face related cost uncertainties.[8] However, Rothkopf (1969) developed a model in which bidders are restricted to bidding a multiple of their cost estimate (rather than an arbitrary function of it), yielding a closed-form solution for equilibrium strategies in two-bidder asymmetric auctions. It is also amenable to finding equilibrium strategies numerically with more than two bidders. This paper uses that model to inform the ongoing debate on policies favoring disadvantaged competitors in procurement and licensing.

Ayres and Cramton (1996) argue that the usual accounting for the costs of subsidizing disadvantaged competitors grossly overestimates them, by neglecting the impact of subsidies on the bidding of unsubsidized competitors. They claim that a subsidy policy can sometimes materially benefit the bidtaker. Our paper lays a theoretical foundation for that claim, and is thus complementary to Ayres and Cramton (1996). They provide an empirical argument that the Federal Communications Commission (FCC), by subsidizing

connections, there might be no reason to think that those bidders started with a disadvantage.

[5] "Designated bidders" is an official term of the Federal Government for bidders granted favorable terms.

[6] This assumption renders McAfee and McMillan's (1989) adaptation of Myerson (1981), which concludes that domestic firms should be favored relative to foreign firms in public procurement auctions, suspect as policy analysis. Our model puts that conclusion on a more realistic footing. Similarly, Bulow and Roberts' (1989) example of subsidizing a disadvantaged bidder is limited to that model. Branco (2002) also uses the independent private-values model.

[7] Cf., Rothkopf (1969), Wilson (1977), Milgrom and Weber (1982).

[8] Asymmetric equilibria have been found for some private-values auction models (and do not exist for some particular symmetric models, cf., Maskin and Riley (2000b)). The dominant strategy in second-price, private-values auctions does not depend on symmetry. Marshall et al. (1994) calculate numerically the independent private-values first-price equilibrium for the example of uniformly distributed types, either two bidders or one bidder facing a cartel. Waehrer (1994), Maskin and Riley (2000a), and Lebrun (1999) have some qualitative characterizations of independent private-values first-price equilibria. Myerson's (1981) model allows for additive corrections to bidders' valuations that depend upon rival valuations, but not for correlated estimating errors. Mares (2001) develops a method for analyzing second-price common-value auctions when the sole source of asymmetry is asymmetrically-sized bidding coalitions and each individual coalition member is symmetrically informed.

minority-owned firms, increased revenue in the 1994 "Regional Narrowband" auction of radio spectrum rights. Their paper argues for the wide applicability of the logic that tilting competitions to favor disadvantaged participants yields the real benefits of advantaged participants competing more aggressively.[9]

The next section outlines the Rothkopf (1969) model and relevant results. After that, the paper uses this model to identify when it is advantageous to a bidtaker to subsidize disadvantaged competitors. It starts with two-bidder situations, and then examines more general cases. The final discussion section considers two potential drawbacks to bidtakers of subsidizing disadvantaged bidders. Might subsidizing inefficient bidders have the undesirable long-run consequence of encouraging entry of inefficient firms? Will the bidtaker lack the information necessary to set a useful subsidy rate? Finally, it discusses when economic efficiency requires subsidization.[10]

2. The Multiplicative-Strategy Model

Asymmetry is inherent in any model of disadvantaged bidders. Game-theoretic modeling of asymmetry in auctions in general has proven difficult.[11] However, there has long been available a class of game-theoretic models that can be solved for asymmetric situations: multiplicative-strategy models (Rothkopf 1969, 1980a).[12] These models restrict bidders' feasible strategies to bidding multiples of their estimates of value or cost rather than arbitrary functions of the estimates: a feasible strategy can be represented by a real scalar, the multiple (or "markup").[13] Thus, a bidder who is using a strategy of bidding 120% of his cost estimate will use that multiple whether his cost estimate is $1,000 or $10,000. Such strategies are often reasonably realistic from a behavioral point of view.[14] They are not generally optimal in game models where any function can be a strategy, but are asymptotically optimal in such models as the amount of prior common information relative to the amount of information contained in the private estimates becomes small (Rothkopf 1980b).[15]

The symmetric version of the model we use is a restricted-strategy variant of the usual common-value auction model. In it, each bidder knows that his cost is the same as every other bidder's. This common cost, c, is unknown when submitting bids. Each bidder draws an independent private sample, $z_i = c\rho_i$, from the same commonly known, unbiased cost-estimating distribution. That is, the relative estimating errors ρ_i are i.i.d. random variables with a mean of 1. Bidders are assumed to use their estimate z_i without any correction for prior information about c. This is equivalent to assuming an (improper) uniform prior on c

[9] Schotter and Corns (1999) also supply supportive evidence from laboratory auctions.

[10] While we refer (conventionally) to a private-sector bidtaker as seeking to minimize expected cost, this is based upon analysis of this auction in isolation. Circumstances will compel some private-sector bidtakers to conduct auctions less myopically, if, say, the number of bidders responds to expected profitability (cf., Harstad (1993)).

[11] Recent surveys of the auction theory literature include Wilson (1977) and Klemperer (1999). Rothkopf and Harstad (1994) offer a critical analysis of it.

[12] Vickrey's pioneering game-theoretic analysis of auctions (1961) considered a model with bidders' private values uniformly distributed on [0,1]. Although he did not restrict his analysis to multiplicative strategies, for this model (specifically, with the lower end of the uniform distribution set at 0), they turn out to be equilibrium strategies.

[13] Finding equilibrium in scalar strategies rather than functions greatly improves the model's tractability; in this case, tractability is enhanced with a relatively reasonable assumption. A number of studies have used multiplicative strategy models of auctions, including decision-theoretic models such as Capen et al. (1971), Curtice and Maines (1973), and Dougherty and Nozaki (1975). Others, including Oren and Rothkopf (1975), Smith and Case (1975), Zinn et al. (1975), Smiley (1979), Wood (1989), Rothkopf (1991), Harstad and Rothkopf (1995), and Fu (1996), are game theoretic. The last five use the model employed in this paper.

[14] Some papers in the scientific literature on bidding models for situations of uncertainty that were authored by individuals with extensive bidding experience in industry use multiplicative-strategy model (Capen et al. 1971, Dougherty and Nozaki 1975). To understand policies it may make sense to assume a restriction typically utilized by bidders, even if it is theoretically suboptimal.

[15] If a bidder has only a vague inkling whether his estimate is high or low, his optimal strategy is unlikely to use that inkling to vary the markup percentage. Rothkopf (1980b) considers a variant of this model, an "affine strategies" model, in which a bidder chooses two parameters, a fixed-bid component and a multiplicative component. He shows that the fixed component becomes a negligible part of the bid as the distribution of estimates becomes more variable.

on $[0,\infty)$. Thus, for any $y > 0$, $\Pr[c < y \mid z_i] = \Pr[\rho_i > z_i/y]$. Each bidder's strategy set consists of multiples $P_i \in \Re^+$, so that i's bid $b_i = P_i z_i$.

The asymmetric version differs only in that the bidders' common knowledge about the bidders' relative true costs is now a ratio different from 1. Thus, for example, it is assumed to be common knowledge that a bidder with a 10% cost disadvantage relative to another bidder will draw his cost estimate from an unbiased distribution that is 10% higher at every probability level.

The multiplicative-strategy model behaves reasonably. It involves an explicit correction for the "winner's curse" inherent in common-value models.[16] Both the equilibrium multiplicative strategies and the resulting profits vary in a rich and qualitatively reasonable way as the number of bidders and the estimating uncertainty vary. The asymmetric case gives a qualitatively sensible effect of cost advantages and disadvantages.

It bears emphasis that the limitation to multiplicative strategies is not an imperfect alternative to a "correct" model, but rather to models that, while more general in selection of a strategy space, impose severe restrictions of their own, including some that it avoids.[17] Thus, it is the use of one approximate model in place of another that employs some different and some matching approximations.

To obtain analytic solutions, Rothkopf's (1969) multiplicative-strategy model uses distributions from extreme-value statistics. For auctions in which bidders are competing to supply, the assumption is that the relative estimating errors are drawn from a two-parameter Weibull distribution. This family of distributions has cumulative function

$$F(\rho) = 1 - \exp(-a\rho^m), \quad \rho > 0$$

and density

$$f(\rho) = am\rho^{m-1}\exp(-a\rho^m), \quad \rho > 0.$$

A shape parameter m and a parameter a that control the scale characterize a member of the family. Each ρ_i is drawn from a distribution with a common (m, a). Below, bidders' uncertainty over project costs is represented by u, the standard deviation of ρ_i. The parameter m determines u: as m goes from 1 to 10 to 100 to ∞, u goes smoothly from 1 to 0.12 to 0.0127 to 0. The exact relationship is

$$u = [\Gamma(1+2/m) - \Gamma^2(1+1/m)]^{-1/2},$$

where Γ is the gamma function given by

$$\Gamma(w) = \int_0^\infty v^{(w-1)} e^v \, dv.$$

(For auctions in which bidders are competing to purchase, the family of distributions is Gumble's (1958) third asymptotic family, which is the distribution of the reciprocal of a Weibull-distributed random variable.)[18]

For the symmetric model in which n risk-neutral bidders are competing to supply, the equilibrium multiplicative strategy is

$$P^* = m(n-1)n^{1/m}/[m(n-1)-1],$$

and the expected value of the equilibrium winning bid is

$$b_{\min} = c + c/[m(n-1)-1], \tag{1}$$

where c is the true but unknown common cost of the job. The expected procurement cost, denoted Z,

[16] The model predates the first academic use of the term "winner's curse" by Capen et al. (1971).

[17] Most notably, this model avoids the assumption of independently distributed private information. Our model may well be more robust to violations of the usual common knowledge assumptions on the structure of the game. All available models depend for hope of analytical tractability on simplifying assumptions. For example, this model shares with most the simplifications of treating the private information of bidders as summarizable via a scalar (see Harstad et al. (1996) and Mares (2001)), and of considering a single auction in isolation (Rothkopf and Harstad 1994 discuss the impact of backing away from this simplification).

[18] For more details on this distribution, see Gumble (1958). Details on the Weibull distribution are available in probability texts such as Olkin et al. (1980) or at ⟨*http://www.itl.nist.gov/div898/handbook/eda/section3/eda3668.htm*⟩. To obtain our formulas for the Weibull distribution and density from the one on the NIST web site, set their location parameter to 0, their shape parameter to m, and their scale parameter to $a^{1/m}$.

Table 1 *n* **Symmetric Bidders**

Precision factor						
u			$n=2$	$n=3$	$n=4$	$n=5$
0.12	Strategy	P_i^*	1.191	1.175	1.188	1.205
	Expected procurement cost	Z	1.111	1.053	1.034	1.026
0.28	Strategy	P_i^*	1.586	1.504	1.543	1.595
	Expected procurement cost	Z	1.333	1.143	1.091	1.067
0.52	Strategy	P_i^*	2.828	2.309	2.400	2.556
	Expected procurement cost	Z	2.000	1.333	1.200	1.143

is simply equal to $b_{\min}$ in this case. Table 1 illustrates symmetric bidding for varying degrees of uncertainty (u) and competition (n bidders). The third bidder adds an incremental level of competition that forces bidders to bid more aggressively—to mark up cost estimates less. In contrast, the predominant impact of a fourth or fifth bidder is to heighten the winner's curse: the low bidder will have the lowest cost estimate of a larger number of estimates, so the bids must be further above cost estimates. Nonetheless, as is clear from Equation (1), the lowest of a larger number of bids is closer to expected project cost.

3. Subsidizing the Less Efficient of Two Potential Suppliers

To introduce asymmetry, let bidder 1 be a "first-line" bidder, meaning that he has access to as efficient a technology as any bidder, and let his unknown cost be c_1. Let bidder 2 be the disadvantaged bidder with cost (before any subsidy) $c_2 \geq c_1$; we will refer to the ratio $d = c_2/c_1$ as the *disadvantage*. The equilibrium strategies for Weibull parameter $m > 1$ are

$$P_1^* = m[1+1/Y(d)]^{1/m}/[m-Y(d)]$$
$$= mX(d)[1+X(d)]^{1/m}/[mX(d)-1]$$

and

$$P_2^* = m[1+1/X(d)]^{1/m}/[m-X(d)]$$
$$= mY(d)[1+Y(d)]^{1/m}/[mY(d)-1],$$

where the simplifying expressions $X(d)$ and $Y(d)$ depend on m:

$$X(d) = \{m(1-d)+[m^2(1-d)^2+4d]^{1/2}\}/2$$

and

$$Y(d) = 1/X(d)$$
$$= \{m(1-1/d)+[m^2(1-1/d)^2+4/d]^{1/2}\}/2.$$

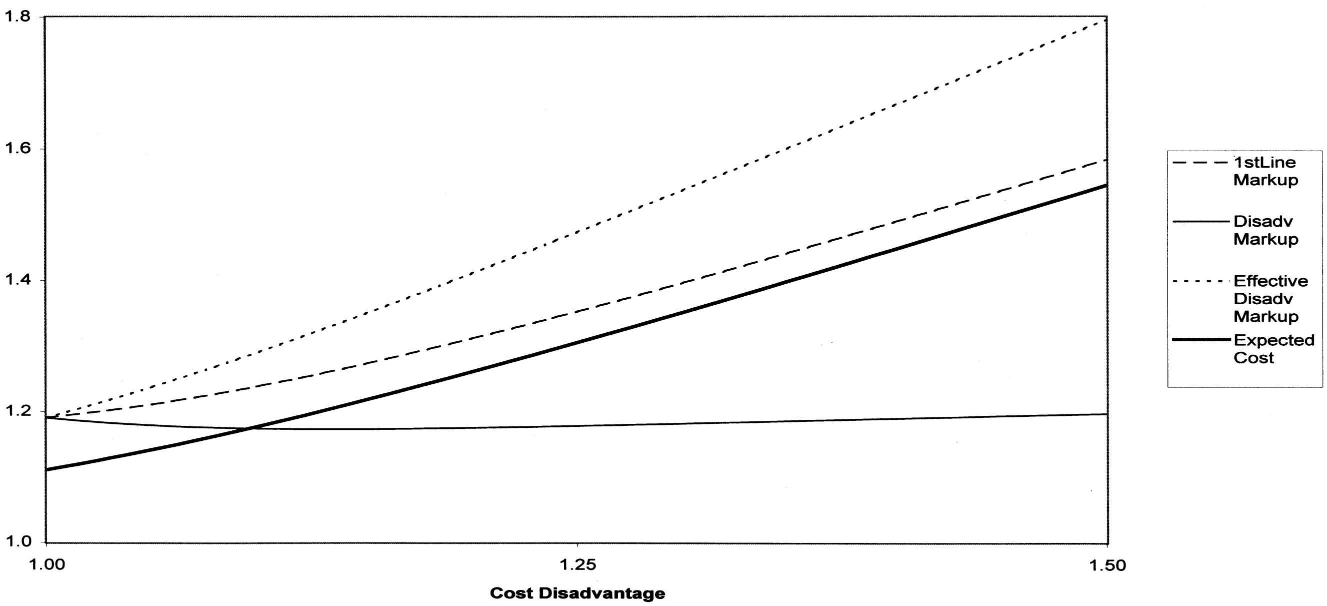
In equilibrium, the probability of bidder 1 being the low bidder is $p_1 = 1/[X(d)+1]$, and the expected payment by the bidtaker is $Z = b_{\min} = md/[m-X(d)]$, in units of c_1.

A bidder with a cost advantage over a single rival uses part of that cost advantage to bid more aggressively and keeps part of it to get a higher profit when he wins. Figure 1 illustrates this for the case $u = 0.12$ used extensively below. As d increases from 1 to 1.5, the disadvantaged bidder 2 continues to use an optimal markup factor P_2^* of close to 1.19, and accordingly his effective markup $d \cong P_2^*$ rises from 1.19 to approximately 1.8. Meanwhile, the optimal markup factor P_1^* of the advantaged bidder 1 rises from 1.19 to approximately 1.6. Thus, the advantaged bidder increases his markup factor by more than half as much as his effective competition, taking advantage of the diminished competition by significantly increasing not only his probability of winning but also his profit if he wins.

Parallel results for auctions in which bidders are competing to purchase are available by replacing costs by values and the parameter m by $-m$. For details on all of these results, see Rothkopf (1969).

Now consider the situation that would result if the bidtaker publicly offers to subsidize the disadvantaged (less efficient) bidder 2, whose cost is $c_2 (> c_1)$. Let the subsidy take the following form: a subsidy set at s implies that, should a disadvantaged bidder make the low bid, he will win the contract but be paid 100s% more than his bid. Because a multiplicative-strategies model with a diffuse prior and Weibull-distributed estimating errors is fully a proportional model, bidder 2 bids as if his cost is reduced proportionately to $c = c_2/(1+s)$. Thus, both bidders bid as if bidder 2's cost is, in fact, c. The disadvantage ratio is now $d = c/c_1 = c_2/[c_1(1+s)]$. Using the above results in equilibrium, the expected payment to the winning bidder, not counting any subsidy, will be $md/[m-X(d)]$. Because the probability that bidder 2 wins is given by $X(d)/[1+X(d)]$, the expected amount of the subsidy is $(c_2-c)X(d)/[1+X(d)]$, and the bidtaker's

Figure 1



overall expected cost is given by $Z = md/[m - X(d)] + (c_2 - c)X(d)/[1 + X(d)]$, in units of c_1.

The results are illustrated in Table 2. As an example, let $u = 0.12$ (column 1 in Table 2 indicates how much uncertainty is present), and let the disadvantaged bidder have a 20% cost disadvantage (thus, column 2 shows $c_2/c_1 = 1.2$). The third row has results for this pair of parameters. In the absence of a subsidy, the more efficient bidder will bid 1.312 times his cost estimate (P_1^*, column 4a), while the disadvantaged bidder will bid 1.175 times his estimate (P_2^*, column 4c). The result is that the disadvantaged bidder wins 32.6% of the time (column 8a), and the expected cost is 1.261 c_1 (Z, column 6a).

If the 20% disadvantaged bidder is fully subsidized, the result is as if his cost were also c_1 (and so d were 1, as in the first row). Each bidder will multiply his cost estimate by 1.191. Each will win half of the time, and the expected payment (before subsidy) to the winner will be only 1.111 c_1. Half of the time, the bidtaker will have to pay an additional subsidy of 20% of this amount. This brings its expected payment up to 1.222 c_1 (not shown in Table 2), more than 3% below the payment with no subsidy.

While better than no subsidy, the full 20% subsidy is more generous than is best for the bidtaker. A subsidy of 14.2% (that is, 71% of a full subsidy, as shown in column 3, now back in the third row) minimizes the bidtaker's expected payments. The disadvantaged bidder rationally responds to a subsidy abating 71% of his disadvantage by increasing his markup on his postsubsidy cost estimate by only 0.28% (from 1.175, column 4c, to 1.179, column 4d), yielding competition for the first-line bidder that is 13.8% more effective.[19] With it, the expected total payment including the subsidy is just 1.215 c_1 (Z, column 6b) The disadvantaged bidder's probability of winning is then 0.445 (column 8b). With the optimal subsidy, expected procurement cost, Z, has fallen 3.67% relative to no subsidy (column 7).

To complete describing Table 2, columns 5a–5d list the expected profit of the first-line bidder (Π_1^*) and disadvantaged bidder (Π_2^*), when the latter is unsubsidized, and then optimally subsidized. For $u = 0.12$, $c_2/c_1 = 1.2$ (still the third row), optimal subsidization decreases Π_1^* from 0.176 c_1 to 0.079 c_1, and increases Π_2^* from 0.02 c_1 to 0.041 c_1.

The entries in Table 2 make clear the pattern of the underlying continuous process. Without a subsidy,

[19] The calculation is $100 * [1.142 * (1.175/1.179) - 1]$. Bidder 1 faces a bid that is 13.8% lower at every probability p.

Table 2 Two Bidders, One of Whom Is Disadvantaged

1	2	3	4a	4b	4c	4d	5a	5b	5c	5d	6a	6b	7	8a	8b
			Strategy				Expected profit				Expected procurement cost			Probability of inefficient outcome	
			First-line P_1^*		Disadvantaged P_2^*		First-line Π_1^*		Disadvantaged Π_2^*		Z				
Precision factor u	Disadvantage factor c_2/c_1	Optimal % of full subsidy	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	% Reduction in exp. cost	(unsub.)	(opt. sub.)
0.12	1.00	—	1.191	1.191	1.191	1.191	0.056	0.056	0.056	0.056	1.111	1.111	—	0	0
0.12	1.05	72%	1.211	1.196	1.179	1.187	0.079	0.061	0.041	0.051	1.142	1.138	0.31%	0.446	0.485
0.12	1.20	71%	1.312	1.212	1.175	1.179	0.176	0.079	0.020	0.041	1.261	1.215	3.67%	0.326	0.445
0.12	1.50	69%	1.583	1.249	1.197	1.174	0.424	0.117	0.010	0.029	1.544	1.348	12.69%	0.221	0.386
0.12	2.00	67%	2.077	1.310	1.222	1.175	0.869	0.175	0.007	0.020	2.040	1.535	24.77%	0.164	0.327
0.28	1.00	—	1.586	1.586	1.586	1.586	0.167	0.167	0.167	0.167	1.333	1.333	—	0	0
0.28	1.05	68%	1.612	1.593	1.564	1.578	0.191	0.174	0.153	0.162	1.368	1.366	0.11%	0.482	0.494
0.28	1.20	67%	1.711	1.617	1.524	1.560	0.274	0.195	0.123	0.151	1.484	1.462	1.51%	0.434	0.479
0.28	1.50	64%	1.968	1.665	1.503	1.537	0.477	0.236	0.094	0.134	1.755	1.640	6.53%	0.368	0.453
0.28	2.00	61%	2.472	1.744	1.510	1.518	0.865	0.301	0.079	0.117	2.253	1.911	15.17%	0.310	0.422
0.52	1.00	—	2.828	2.828	2.828	2.828	0.500	0.500	0.500	0.500	2.000	2.000	—	0	0
0.52	1.05	62%	2.882	2.848	2.779	2.809	0.532	0.512	0.494	0.498	2.051	2.050	0.05%	0.494	0.498
0.52	1.20	60%	3.056	2.906	2.664	2.760	0.632	0.545	0.482	0.492	2.209	2.195	0.64%	0.477	0.491
0.52	1.50	58%	3.441	3.013	2.529	2.688	0.850	0.607	0.473	0.484	2.549	2.472	3.02%	0.451	0.481
0.52	2.00	54%	4.152	3.176	2.426	2.610	1.244	0.700	0.488	0.477	3.155	2.904	7.96%	0.423	0.468

the disadvantaged bidder presents little competition for the first-line bidder. Accordingly, the first-line bidder rationally bids a significant markup on his cost estimate, one that allows the disadvantaged bidder a significant chance of winning. As the subsidy is increased, the first-line bidder becomes more aggressive, but the disadvantaged bidder's probability of winning continues to increase until it reaches 50% when the subsidy makes him an equal competitor. The basic intuition is this: it does not pay the first-line bidder to bid aggressively enough to shut out the disadvantaged bidder; driving his own winning probability to 1 reduces too substantially his profitability in the events he would already win. Without any subsidy, the expected cost to the bidtaker is slightly above the disadvantaged bidder's cost. When the bidtaker offers a subsidy to the disadvantaged bidder, he is forcing the first-line bidder to bid more aggressively. The savings this produces more than offsets the expected cost of the subsidy.[20] It is noteworthy that the optimal subsidy produces only a modest increase from 32.6% to 44.5% in the probability that the inefficient bidder will win.

When the uncertainty over project costs (i.e., u) increases, the results are qualitatively similar. A subsidy that partially reduces the disadvantage is preferred to a full subsidy, which, in turn, is preferred to no subsidy. With greater uncertainty, the first-line bidder is less equipped to take advantage of his technological position in the bidding, so even without a subsidy, the disadvantaged bidder wins more frequently at higher u. Accordingly, the optimal subsidy is somewhat smaller, changes the probability of winning somewhat less, and produces less economic impact in terms of percentage reduction in Z.

The analytic formula for the optimal subsidy with two bidders is complex. However, the desirability of some partial subsidy holds whenever the disadvantage is significant. The following proposition is proven in the Appendix, using $c_1 = 1$ as a normalization.

Proposition. *That c_2 exceed a threshold value, $1+4/(m-1)^2$, is sufficient for there to exist some partial subsidy s^* (dependent on u and c_2, $1 < 1+s^* < c_2$) such that any subsidy level $s > s^*$ has a higher expected procurement cost than s^*.*

For the principal example discussed above, $u = 0.12$, the proposition applies whenever $c_2/c_1 > 1.049$. This sufficient condition is not necessary; calculations show that for $c_2/c_1 = 1.02$, a partial subsidy is optimal.

4. Subsidizing with More Than Two Suppliers

This section uses the n-bidder version of the multiplicative-strategy model to explore similar issues to those discussed in the context of two bidders. Section 5 of Rothkopf (1969) discusses the n-bidder model. It does not have a known analytic equilibrium solution; hence, we rely on numerical methods. Because bidders' strategies are modeled as scalars, this is not as daunting as it would be if they were general functions. This approach, while adequate, does not provide the same precision that the analytic formulas in the two-bidder analysis did. Instead of adjusting the subsidy arbitrarily finely to find precise minima, we consider only subsidies that change the ratio of the two bidders' cost to the one asymmetric bidder's cost by a multiple of 0.01. We then approximate minima by interpolation.

4.1. A Single Disadvantaged Bidder

If there are two or more first-line bidders, the situation of the disadvantaged bidder is tenuous. If his disadvantage (after subsidy) is too great, he has

[20] The reduction in expected cost results from the increased aggressiveness that the first-line bidder faces, and his consequent reduction in his own markup. The effect of the subsidy on the disadvantaged bidder is a secondary issue. His multiple on his postsubsidy cost estimate may be larger than his unsubsidized markup, although for a sensible subsidy, it never increases in percentage terms more than the subsidy; subsidizing leads to a lower markup for cases where the disadvantage remains large and the estimating uncertainty small. Interestingly, when the disadvantage is extreme, the disadvantaged bidder's expected profit can actually be reduced by increasing his subsidy. Sufficiently large disadvantages and large cost uncertainties lead to bidder 1 effectuating such monopoly power from so little competition that bidder 2's expected profit is increasing in the disadvantage (this can be seen in the two bottom entires in column 5c) and, consequently, bidder 2 is made worse off via an optimal subsidy (columns 5c and 5d, bottom row).

no profitable bid. The first-line bidders compete with each other and drive the expected price below one that he can afford. If he is subsidized sufficiently to compete, he will still have only a secondary effect because the other bidders will already be competing with each other. However, it still can benefit a bidtaker to subsidize a bidder who is not too disadvantaged.

Table 3 presents for the case of two first-line bidders results corresponding to those in Table 2. Considering our usual first case, $u = 0.12$, $c_2/c_1 = 1.2$, the one disadvantaged bidder, if unsubsidized, cannot profitably make any bid with a positive probability of winning. Indeed, no subsidy less than 7% has any effect, as it still does not bring him into competition. It does pay the bidtaker to bring this third bidder into competition; the optimal subsidy is approximately 12.1%, 61% of a full subsidy, or about 6/7 as much of a subsidy as was optimal with but one first-line bidder. Here, though, the subsidy makes relatively less difference, offering hardly any expected profit to the disadvantaged bidder, who wins less than one time in eight. The reduction in cost from subsidizing is but 1/4 that of the two-bidder case.

With $u = 0.12$, there is a limit to how far a bidtaker should be willing to go to help the disadvantaged bidder who may become a third competitor. The fourth row shows that a bidder at a 50% disadvantage cannot profitably be brought into play: a subsidy less than 67.8% has no impact, still yielding an auction only between the two first-line bidders; any larger subsidy costs more than the advantage arising from the slight increase in first-line bidding aggressiveness it creates. Note, however, that a similarly 50%-disadvantaged bidder is worth subsidizing when estimating uncertainties are higher; for $u = 0.52$, the high uncertainty makes adding a third competitor sufficiently important that the percentage gain from subsidizing exceeds that of the two-bidder case.

The examples we have examined suggest that offering any subsidy to bring in a single disadvantaged bidder as the fourth competitor for three first-line bidders does not pay.

4.2. One First-Line Bidder

Next, we turn to the situation in which there are more than two bidders, all but one of whom is disadvantaged. This models the situation in which one bidder has a natural advantage—perhaps incumbency, natural monopoly, or critical mass—that all others lack. The model grants the same subsidy to any subsidized bidder; the one first-line bidder is facing stiffer competition in that he must outbid the most competitive of these bidders to win.

As the corresponding Table 4 shows, subsidies are now economically more important than when only one disadvantaged bidder could be assisted. The optimal subsidies are generally similar in magnitude to those in the two-bidder case, even though two competitors are being offered subsidies, and a subsidy gets paid notably more often.

5. Additional and Concluding Considerations

5.1. Allocative Efficiency Considerations

As modeled above, the bidtaker ignores the allocative inefficiency that arises when the disadvantaged bidder wins and then incurs higher costs to fulfill the same contract than the first-line bidder would have incurred. For many contexts, this focus merely on expected project costs is too narrow. Because tax revenues have efficiency costs, a public-sector bidtaker should be concerned both about project cost and allocative inefficiency. In private-sector procurement as well, bidtakers who have to attract bidders may well be concerned with some combination of cost and efficiency (see Harstad 1990, 1993). In dynamic settings where a bidtaker is repeatedly procuring from the same bidders, having a given level of procurement cost translate into higher supplier profitability through reduced inefficiencies may improve the long-run competition to supply.

Small levels of subsidy can generate allocative inefficiencies that are secondary relative to the cost savings of more aggressive bidding by first-line competitors. While details vary with the number of bidders, the underlying degree of uncertainty, and the particular way in which project cost and resultant inefficiencies compare in the objective function, concerns with efficiency often serve only to reduce somewhat the best level of subsidy. Thus, efficiency-conscious bidtakers still gain by offering subsidies to

Table 3 Three Bidders, One of Whom Is Disadvantaged

1	2	3	4a	4b	4c	4d	5a	5b	5c	5d	6a	6b	7	8a	8b
			Strategy				Expected profit				Expected procurement cost			Probability of inefficient outcome	
			Both first-line P_1^*		Disadvantaged P_2^*		Both first-line Π_1^*		Disadvantaged Π_2^*		Z				
Precision factor u	Disadvantage factor c_2/c_1	~Optimal % of full subsidy	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	% reduction in exp. cost	(unsub.)	(opt. sub.)
0.12	1.00	—	1.175	1.175	1.175	1.175	0.018	0.018	0.018	0.018	1.053	1.053	—	0	0
0.12	1.05	59%	1.175	1.174	1.212	1.183	0.030	0.022	0.004	0.011	1.071	1.069	0.21%	0.183	0.275
0.12	1.20	61%	1.191	1.178	—	1.252	0.056	0.037	0.000	0.002	1.111	1.101	0.91%	0	0.122
0.12	1.50	—[a]	1.191	1.191	—	—	0.056	0.056	0.000	0.000	1.111	1.111	0	0	0
0.12	2.00	—[b]	1.191	1.191	—	—	0.056	0.056	0.000	0.000	1.111	1.111	0	0	0
0.28	1.00	—	1.504	1.504	1.504	1.504	0.048	0.048	0.048	0.048	1.143	1.143	—	0	0
0.28	1.05	59%	1.502	1.503	1.522	1.509	0.059	0.052	0.032	0.041	1.164	1.161	0.19%	0.281	0.312
0.28	1.20	61%	1.527	1.503	1.753	1.536	0.107	0.064	0.005	0.027	1.243	1.209	2.72%	0.122	0.163
0.28	1.50	59%	1.586	1.516	—	1.654	0.167	0.092	0.000	0.010	1.333	1.278	4.15%	0	0.205
0.28	2.00	—[c]	1.586	1.586	—	—	0.167	0.167	0.000	0.000	1.333	1.333	0	0	0
0.52	1.00	—	2.309	2.309	2.309	2.309	0.111	0.111	0.111	0.111	1.333	1.333	—	0	0
0.52	1.05	59%	2.311	2.310	2.316	2.310	0.123	0.116	0.095	0.105	1.356	1.355	0.09%	0.311	0.324
0.52	1.20	61%	2.335	2.313	2.409	2.322	0.164	0.128	0.058	0.089	1.434	1.416	1.24%	0.246	0.302
0.52	1.50	83%	2.463	2.326	2.984	2.373	0.270	0.145	0.016	0.066	1.623	1.521	6.29%	0.132	0.302
0.52	2.00	53%	2.828	2.371	—	2.551	0.500	0.199	0.000	0.038	2.000	1.657	17.14%	0	0.201

[a] Any subsidy less than 67.8% of a full subsidy is optimal.
[b] Any subsidy less than 88.9% of a full subsidy is optimal.
[c] Any subsidy less than 50.4% of a full subsidy is optimal.

Table 4 Three Bidders, Two of Whom Are Disadvantaged

1	2	3	4a	4b	4c	4d	5a	5b	5c	5d	6a	6b	7	8a	8b
			Strategy				Expected profit				Expected procurement cost			Probability of inefficient outcome	
			First-line P_1^*		Both disadvantaged P_2^*		First-line Π_1^*		Both disadvantaged Π_2^*		Z				
Precision factor u	Disadvantage factor c_2/c_1	~Optimal % of full subsidy	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	(unsub.)	(opt. sub.)	% reduction in exp. cost	(unsub.)	(opt. sub.)
0.12	1.00	—	1.175	1.175	1.175	1.175	0.018	0.018	0.018	0.018	1.053	1.053	—	0	0
0.12	1.05	58%	1.181	1.174	1.184	1.178	0.041	0.026	0.010	0.014	1.093	1.087	0.60%	0.540	0.612
0.12	1.20	64%	1.285	1.186	1.217	1.186	0.125	0.046	0.004	0.009	1.231	1.176	4.48%	0.348	0.518
0.12	1.50	67%	1.574	1.261	1.261	1.200	0.274	0.079	0.001	0.006	1.535	1.320	13.97%	0.223	0.427
0.12	2.00	64%	2.055	1.297	1.295	1.220	0.425	0.133	0.001	0.003	2.018	1.513	25.02%	0.164	0.337
0.28	1.00	—	1.504	1.504	1.504	1.504	0.048	0.048	0.048	0.048	1.143	1.143	—	0	0
0.28	1.05	58%	1.505	1.502	1.510	1.506	0.067	0.055	0.039	0.044	1.184	1.180	0.34%	0.616	0.646
0.28	1.20	70%	1.574	1.505	1.539	1.510	0.129	0.067	0.024	0.039	1.317	1.287	2.33%	0.509	0.616
0.28	1.50	61%	1.811	1.541	1.594	1.528	0.238	0.106	0.013	0.028	1.593	1.477	7.26%	0.402	0.542
0.28	2.00	58%	2.316	1.617	1.657	1.552	0.373	0.154	0.008	0.020	2.101	1.755	16.45%	0.323	0.479
0.52	1.00	—	2.309	2.309	2.309	2.309	0.111	0.111	0.111	0.111	1.333	1.333	—	0	0
0.52	1.05	58%	2.316	2.310	2.311	2.310	0.129	0.118	0.100	0.107	1.381	1.377	0.26%	0.645	0.658
0.52	1.20	58%	2.389	2.322	2.327	2.313	0.178	0.136	0.079	0.097	1.526	1.505	1.34%	0.592	0.636
0.52	1.50	58%	2.641	2.362	2.373	2.322	0.259	0.165	0.057	0.084	1.822	1.745	4.25%	0.527	0.604
0.52	2.00	57%	3.222	2.459	2.444	2.342	0.363	0.206	0.041	0.070	2.357	2.114	10.30%	0.465	0.567

disadvantaged competitors, often ones that are a sizable fraction of the subsidy level that would minimize expected procurement cost.

5.2. Long-Run Effects of Subsidies

The results presented have been for a one-time foray into procurement auctions, with exogenous numbers of first-line and disadvantaged bidders. Firms or agencies that repeatedly procure may face additional impacts of subsidizing disadvantaged bidders. While a model of entry into the positions of first-line and disadvantaged bidders is beyond the scope of this paper, we believe that partial subsidies will often remain a sensible policy. Subsidies can be seen in Tables 2–4 to enhance the profitability of disadvantaged bidders, and this enhances incentives for a disadvantaged bidder to enter the business of bidding to supply these contracts. However, comparing Table 3 values for Π_1^* with Table 4 values for Π_2^* (both "optimally subsidized"), it is never the case that a firm with a choice of becoming a first-line or a disadvantaged bidder would choose the latter. Suppose that a bidtaker could ensure by a permanent commitment to a "no subsidies" policy that a second first-line bidder would immediately enter into competition, while a subsidy policy would, instead, have led to the third and final bidder to enter being a disadvantaged bidder. In that event, a comparison of the unsubsidized Z values in Table 3 with the optimally subsidized Z values in Table 4 indicates that the assumed impact on entry favors the "no subsidies" policy. That supposition, though, especially the immediacy of entry, strikes us as unlikely in many circumstances. We have calculated that increasing somewhat the likelihood that an eventual entrant is a disadvantaged bidder probably does not completely undo the advantages of subsidies discussed above.

5.3. Imperfectly Informed Bidtakers

The model presented assumes that both first-line and disadvantaged bidders know the disadvantage ratio, c_2/c_1. However, the analytics require no presumption that the bidtaker knows c_2/c_1. In some instances, as when a private-sector developer repeatedly auctions contracts to install sewer pipes in new subdivisions, experience may lead the bidtaker to know this variable nearly as well as the bidders. Sometimes, however, the bidtaker may be less well informed. In particular, a firm or public-sector agency that procures a variety of goods and services from a variety of industries may well be relatively uninformed (compared to the bidders whose livelihoods depend upon knowing industry conditions) about cost differentials in each of the industries in which it deals. So it is sensible to ask whether a subsidy chosen by a bidtaker with somewhat incorrect beliefs about c_2/c_1 might do more harm than good. Our results show sufficient robustness that cautious choice of a subsidy is quite likely to be better than no subsidy at all. While optimal subsidies generally run 55% to 75% of full subsidies, we have observed that any subsidy level less than 70% of a full subsidy will be an improvement over no subsidy for a broad range of actual values of c_2/c_1. Generally, a subsidy that turns out to be somewhat low still captures the bulk of the available reduction in expected procurement cost, and a subsidy that is moderately higher still results in lower expected procurement costs than no subsidy. Thus, even a bidtaker who is quite uncertain about the extent of the disadvantaged bidder's handicap is better off offering some conservatively small subsidy.

It is also plausible that the bidtaker may have incorrect beliefs about the amount of estimating uncertainty bidders face. The optimal subsidy is also rather insensitive to u.

5.4. Subsidies, Yes; Set-Asides, No

Briefly summarizing, partially subsidizing disadvantaged bidders, generally, more than compensates for the cost of the subsidy due to increased aggressiveness by first-line bidders. Preferred subsidies become larger when the bidtaker has more need for competition from the disadvantaged bidders, especially when the cost disparity is large. Additional disadvantaged bidders reduce somewhat the optimal subsidies. An additional first-line bidder makes this effect much more pronounced, and two additional first-line bidders generally eliminate the advantage of subsidies.

We close by pointing to distinctly different economic consequences of two affirmative action policies that often go hand-in-hand: subsidies and set-asides for disadvantaged bidders. In the FCC's Regional

Narrowband auction, minority-owned firms were favored by (1) setting aside two of the six licenses in each geographic region for bidding only by them, (2) subsidizing their bids on all licenses, and (3) subsidizing their bids on the set-aside licenses to a greater degree. Ayres and Cramton (1996) treat this as a single policy and analyze its effects. However, the revenue gains from the policy that they find when they trace through the bidding are the result of (2) above, viewed as a separable policy. The evidence from that auction is entirely consistent with the hypothesis that revenue would have been even higher had the FCC set up subsidies for minority bidders at the two levels used, but then allowed all bidders to bid on any of the licenses. The major impact of (1) above was to prevent unsubsidized bidders from competing for some of the licenses. Indeed, it was because disadvantaged bidders were not satisfied with competing for the set-aside licenses that they generated more aggressive bidding by unsubsidized firms.

A subsidy policy affects only probabilistically the proportion of contracts awarded to disadvantaged bidders. Detailed knowledge of industry conditions is needed to calculate in advance the probabilities that will result from any particular level of subsidy. One view of the FCC's policies is that they interpreted a congressional mandate as meaning that they had to guarantee that disadvantaged bidders would win at least one-third of the licenses. Set-asides make such guarantees straightforward, while they would add considerable complications to an auction where all bidders can, in principle, compete for all assets.[21]

A complete model of entry by potential first-line and disadvantaged bidders into multiple-unit auctions with subsidies and set-asides is beyond the scope of this paper. However, it is clear that set-asides remove or, at least, reduce the incentive for disadvantaged bidders to compete with first-line bidders, while subsidies enhance this incentive. It is the natural best response of first-line bidders to this competition, more aggressive bidding, that often serves to more than cover the costs of subsidies. This reaction by first-line bidders is completely ignored in the usual accounts of the cost of subsidy policies. In fact, as we have shown, the "cost" of subsidies may often be negative.

[21] It is, however, possible to accomplish a guarantee without prespecified set-asides. For example, the FCC's multi-stage progressive auction could be conducted with the rule that the tentatively winning bids at the end of any round are the combination of bids that yield the greatest revenue among all combinations that award, at least, the requisite number of licenses to disadvantaged bidders. If the high bids would award too few licenses to them, then this rule implies that the bids by disadvantaged bidders that come closest to the high bids on some licenses are declared tentative winners. The shadow price on this constraint implicitly defines the subsidy.

Acknowledgments

The authors gratefully acknowledge research support from the National Science Foundation. Harstad thanks the Fuqua School of Business, Duke University, for research support and hospitality while this paper was being revised. The authors thank, without implicating, Peter Cramton, Aleksandar Pekec, and three referees for helpful comments, and Robert Nau for extensive help thoughtfully guiding this paper toward its present form.

Appendix

Preliminary considerations: by normalizing $c_1 = 1$, $d = c$, so $Mc/Md = 1$.

Proof of Proposition. It is clear that subsidizing more than fully, to the point where bidder 2 has a cost advantage, raises expected procurement costs relative to a full subsidy. It suffices for the proposition, then, to demonstrate that Z is falling as d is increased differentially when evaluated at $d = 1$, for c_2 sufficiently large.

$$X(d) = \{m(1-d) + [m^2(d-1)^2 + 4d]^{1/2}\}/2,$$

so

$$X' = \{[m^2(d-1) + 2]/[m^2(d-1)^2 + 4d]^{1/2} - m\}/2.$$

(The notation (d) as an argument of X and X' is dropped below when clear.) Then $X'|_{d=1} = -(m-1)/2$ and $X(1) = 1$. Because

$$\begin{aligned} Z &= md/[m - X(d)] + (c_2 - c)X(d)/[1 + X(d)], \\ Z_d &= [m(m-X) + mdX']/(m-X)^2 \\ &\quad + \{(1+X)[(c_2-c)X' - X] - (c_2-c)XX'\}/(1+X)^2 \\ &= [m(m-X) + mdX']/(m-X)^2 - X/(1+X) + (c_2-c)X'/(1+X)^2. \end{aligned}$$

Because $d = 1 \Rightarrow c = c_1 = 1$,

$$Z_d|_{d=1} = [m/2]/(m-1) - 1/2 - (c_2-1)(m-1)/8, \qquad \text{(A1)}$$

which (because $m > 1$) for given c_1 is clearly negative for large enough c_2. Letting c_2^0 be the value of c_2 for which (A1) is zero gives

$$c_2^0 = 1 + 4/(m-1)^2$$

as large enough c_2, since subsidizing to the point where bidder 2 has an effective cost advantage is clearly suboptimal. Q.E.D.

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Accepted by Robert Nau; received December 1997. This paper was with the authors 29 months for 4 revisions.