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Games, Econometrics and Optimisation

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Chapter 8

PERFECT EQUILIBRIA OF SPECULATIVE FUTURES MARKETS

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1. Introduction

Barriers to investing in futures markets are negligible. This may be the reason why futures markets are typically modeled as perfectly competitive. The underlying industry which produces the commodity traded on a futures market, however, is often heavily concentrated, for at least some stages of the production process.[1] A producer with oligopolistic influence upon the time path of production, and thus upon the market value of futures contracts a maturity, is never a negligible actor in the futures market; purely competitive models cannot capture his rational futures contracting activity. Behaviour ascribed to non-producing speculators in competitive models becomes myopic, even perilous, when the other party to a futures contract they sign subsequently acts to alter the cash price, i.e., the spot price at maturity (as a direct result of signing the contract).

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[1] A few independent actors have substantially influenced production and prices in such exhaustible resource markets as copper, tin and petroleum. Intermediate stages of the production and distribution process appear oligopolistic for some agricultural products traded on futures markets.

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When speculators and producers view futures prices and cash prices as immutable, a model of their individual behaviour need not take into account institutional procedures by which prices are determined on futures markets. The particular price-setting (or, more generally, allocation-reaching) rules being used may nonetheless play a key role in describing the strategic behaviour of actors with nonnegligible influence on prices.

A third sense in which the standard approaches to futures market modeling may be limiting their scope is a tendency to focus on futures contracts as a means of hedging against price risk in the spot market at maturity.[2] Upon casual observation, we are inclined to model the bulk of futures transactions as purely speculative, with any relation to hedging of underlying stocks secondary.[3] In essence, many futures contracts seem to be bets about the level of the spot price at maturity, with each party undaunted by the willingness of the other to accept the bet.

We offer an alternative model of a futures market for an exhaustible resource, based on an underlying oligopolistic industry structure. It features a stylized set of price-determining rules, and agents who knowingly hold divergent beliefs about the economic value of the resource at maturity. As our purpose is demonstrative, the model is in many directions simple, almost stark. Nonetheless, complex interplay arises among contractual behaviour, rules, and anticipation of behaviour to follow.

With an underlying oligopoly, inefficient allocation of resources attained in subgame-perfect equilibrium may have been anticipated. The nature of the inefficiency is insufficient volume of futures market activity; we discuss its relation to incentives facing market-making institutions who determine the trading rules.

The futures market examined here is streamlined by having one type of contract, which technically states that, in return for a mutually agreeable, certain payment, the seller will provide the buyer with an agreed quantity of the resource at the one and only maturity date (which we call period 2). However, the contract in fact specifies that, once the cash price at maturity becomes known, the buyer pays the seller the contracted futures price minus the cash price (be this difference positive or negative). Thus futures positions

[2] Several models and extensive references are in Anderson (1984).

[3] Transactors' positions are closed out day-by-day, including at maturity, via adjustments to balances in margin accounts. Delivery at maturity is almost unheard of in futures markets.

are not constrained by the total amount of the resource available; nor are they subject to any budget constraints.[4]

The model enriches this simple futures market with explicit attention to the institutional rules by which such futures markets as the London Metal Exchange operate. A stylized version of their "open cry" auction, with its incentives for consummating trades just before the closing bell, is built in Sections 2 and 3 below.

Attention is focused on the economic impact of rules which have one party to a contract, the offeror, determine a price and a maximum quantity, and then have the other party, the acceptor, determine the actual quantity exchanged. Thus, the offeror fixes the price, the acceptor fixes the quantity. This is the source of inefficient subgame-perfect equilibrium allocations, because it puts the offeror in a temporary monopoly position.

The resource-extracting industry is presented in simplest form, a Cournot (quantity-setting) duopoly. We will refer to the actors as producers $\mathscr{A}$ and $\mathscr{B}$, but abstract from all production decisions except the choice to influence profitability of futures contracts. That is, each producer will have an exogenously determined stock of resources, and all producers simultaneously decide how much of their stock to supply prior to maturity (period 1), the remainder being supplied at maturity (period 2). This decision is made after the futures market closes, but before revelation of the uncertain level of final demand for the resource.

No explicit role is given to brokers; we limit our futures market analysis to the two producers $\mathscr{A}$ and $\mathscr{B}$, one of whom must be risk-averse, and a single, risk-neutral speculator $\mathscr{S}$.[5] These three hold inconsistent prior beliefs about the level of final demand for the resource. These beliefs are common

[4] Any budget constraint in the model would likely be more severe than budget constraints seen in futures markets, except when a crash is fresh in memory. It is more illuminating to derive finite volumes of futures trades between producers and speculators from producers' manipulative opportunities.

[5] Many speculators competing in the terms of trade offered to the two producers on the futures market would primarily serve to enhance the producers' manipulative opportunities. The already cumbersome notation needed to keep track of contract offers and acceptances would explode in complexity with the added speculators. If competition between speculators sufficiently reduced the level of profit expected from trading a given volume of contracts, and a speculator did not know whether he was trading with a producer (and what net short position that producer has attained in other futures contracts), subgame-perfect equilibrium futures trading might collapse with many speculators. None of the analysis depends on the single speculator being risk-neutral. The model can only accommodate two risk-neutral actors if exactly one of them is a producer with a nonzero initial resource stock; otherwise futures positions become infinite.

knowledge. The situation is thus analyzed as a two-stage noncooperative game with inconsistent incomplete information.[6] The futures game is positive-sum (in terms of expected profit) whenever positions taken in the futures market are sufficiently small relative to differences in the prior beliefs.

2. The game

In the futures market, an *offer* is a specification of (i) whether the offeror is suggesting that he *buys* (in the jargon of futures markets, he "goes *long*") or *sells* ("goes *short*"), (ii) a *price* quotation, and (iii) a *maximum quantity*, in units of some "standard contract" as specified by the market-maker. An *acceptance* is a specification of (i) which offer is being accepted, and (ii) the *quantity* (the *position*) being accepted. An acceptance is constrained to be at most the maximum quantity offered, less any quantity of the same offer already accepted by another player. A *rejection* can be thought of as the special case where the accepted position is zero.

The game proceeds as follows.

Step 1. The futures market opens. Simultaneously, each player publicly and irrevocably announces one offer to buy and one offer to sell, each with a price and a maximum quantity.[7]

Step 2. Nature chooses a commitment order, a permutation of $(\mathscr{A}, \mathscr{B}, \mathscr{S})$, each drawn with equal probability.

Step 3. The player designated *first* in the commitment order irrevocably accepts or rejects each of the four offers made by another player.

Step 4. The player designated *second* in the commitment order irrevocably accepts or rejects each of the four offers made by another player.

Step 5. The player designated *third* in the commitment order irrevocably accepts or rejects each of the four offers made by another player. The futures market then *closes.*[8]

[6] Selten (1982, Sections B.9–B.13), formalizes noncooperative games with inconsistent incomplete information. See van Damme (1987) for a definition and discussion of the restriction to subgame-perfect equilibria.

[7] It will always be possible to announce one of these offers at a price guaranteeing rejection.

[8] Intermediate steps before or after Step 2 could have been added, where players may irrevocably accept offers made by other players, but rejections during these intermediate steps would be tentative. Such steps would not alter the set of subgame-perfect equilibria. An alternative set of rules would have the players simultaneously make irrevocable acceptances or rejections just before the futures market closes. This would yield at least a 3-parameter family of subgame-perfect equilibria, with little qualitative features in common. Since this is not a signaling game, further refinements would not notably reduce the set of equilibria. Equilibrium selection theory (Harsanyi and Selten, 1988) would obtain a unique outcome, but a cooperative element would be de facto introduced, and all relationships between trading rules and behaviour would disappear.

Step 6. The ***extraction subgame*** occurs: producers $\mathscr{A}$ and $\mathscr{B}$ simultaneously decide what fraction of their exogenous stock of the resource to supply before maturity of the futures contracts. Technically, this ends the game.

Step 7. The aggregate level of final demand for the resource (both before and at maturity) is revealed. Together with the decisions made in Step 6, this determines the spot price at maturity.

The payoff to each player is the certainty equivalent level of profit; notation to specify payoff functions is introduced in the next section.

3. Formalisms

The analysis requires notational conventions that keep track of the identity of the player suggesting a contract, the player accepting, and the direction of flow (which player goes short), all while striving for coherence. The cash market will have a uniform price at each date, represented by p_t, $t = 1, 2$. A superscripted price is a futures price (sometimes called a "forward" price), as suggested by the agent shown, i.e., $\boldsymbol{p}^a$ is the price at which $\mathscr{A}$ offers to buy futures, to go long. The convention of superscripting the offeror is maintained throughout.

Typestyle distinguishes short sales from long sales, in a manner that is hopefully mnemonic. $\mathscr{A}$ offers to buy at price $\boldsymbol{p}^a$ (bold typeface for "buy"), and to sell at price p^a (soft typeface for "sell"). Each offer combines a price with a maximum position (quantity), as in $(\boldsymbol{p}^s, \boldsymbol{f}^s)$, (p^s, f^s), $\mathscr{S}$'s offers to go long and short, respectively. Since typestyle indicates the direction of a trade, all positions are nonnegative numbers.

Players accepting contracts are identified throughout by subscripts. The amount of offer (p^s, f^s) which $\mathscr{A}$ accepts is represented as f^s_a, which is a long position for $\mathscr{S}$, and thus a short position for $\mathscr{A}$. Correspondingly, (p^s, f^s_a) is the agreed terms where $\mathscr{S}$ set the price, $\mathscr{A}$ determined the position, and $\mathscr{A}$ goes long. For example, $\mathscr{B}$ offers to go short in (p^b, f^b), $f^b_a \leq f^b$ is the extent to which $\mathscr{A}$ agrees to accommodate by going long. $\mathscr{B}$'s revenue on this contract is then $p^b f^b_a$.

The six offers announced allow up to twelve contracts to be formed (each player can accept at most four offers). Positions on different contracts cancel each other (but not in revenue effects, unless prices match), so payoffs depend upon net short positions, which are:

$$\mathbf{A} := f^a_b + f^a_s - \boldsymbol{f}^a_b - \boldsymbol{f}^a_s + \boldsymbol{f}^b_a - f^b_a + \boldsymbol{f}^s_a - f^s_a,$$
$$\mathbf{B} := f^b_a + f^b_s - \boldsymbol{f}^b_a - \boldsymbol{f}^b_s + \boldsymbol{f}^a_b - f^a_b + \boldsymbol{f}^s_b - f^s_b$$

and

$$\mathbb{S} := f_a^s + f_b^s - f_a^s - f_b^s + f_s^a - f_s^a + f_s^b - f_s^b .$$

A position which enters positively in net short position also enters positively in net revenue on the futures market, which is:

$$R^a := p^a(f_b^a + f_s^a) - p^a(f_b^a + f_s^a) + p^b f_a^b - p^b f_a^b + p^s f_a^s - p^s f_a^s ,$$

$$R^b := p^b(f_a^b + f_s^b) - p^b(f_a^b + f_s^b) + p^a f_b^a - p^a f_b^a + p^s f_b^s - p^s f_b^s$$

and

$$R^s := p^s(f_a^s + f_b^s) - p^s(f_a^s + f_b^s) + p^a f_s^a - p^a f_s^a + p^b f_s^b - p^b f_s^b .$$

The two producers have exogenous initial stocks of the resource s_{a0} and s_{b0}, these levels are common knowledge. Producer $i = \mathcal{A}, \mathcal{B}$ extracts and sells q_{i1} units of resource in the single cash market that occurs prior to maturity of futures contracts, and extracts and sells

$$q_{i2} = s_{i0} - q_{i1}, \quad i = \mathcal{A}, \mathcal{B}, \tag{1}$$

units in the single cash market at maturity. For simplicity, both extraction costs and the rate of interest are zero.

Each cash market will exhibit the same instantaneous inverse market demand curve:

$$p_t(q_{at} + q_{bt}) = \frac{\alpha}{\beta} - \frac{q_{at} + q_{bt}}{\beta}, \quad t = 1, 2, \tag{2}$$

where $\alpha > 0$ is unknown, and $\beta > 0$ is common knowledge.[9] Let

$$\hat{p} = \frac{\alpha}{\beta} - \frac{s_{a0} + s_{b0}}{2\beta};$$

$\hat{p}$ is the price that would prevail in both cash markets in Cournot equilibrium, were there no futures market activity by producers. As such, $\hat{p}$ is a natural benchmark. It is more convenient to express uncertainty in terms of $\hat{p}$ than α; this will be done throughout. Each player will have expectations (i.e., prior beliefs) about $\hat{p}$.

[9] This formulation simplifies by omitting any speculative demand for stocks. Were speculative demand included, it would act as a multiplier effect on the manipulative possibilities producers have in this model.

Finally, the payoff functions can be specified. For each player, payoff is the certainty equivalent level of profit:

$$C^a = E_a[p_2(q_{a2}+q_{b2})](q_{a2}-\mathbf{A})+p_1(q_{a1}+q_{b1})q_{a1}+R^a-\tfrac{1}{2}K_a(s_{a0}-\mathbf{A})^2,$$

$$C^b = E_b[p_2(q_{a2}+q_{b2})](q_{b2}-\mathbf{B})+p_1(q_{a1}+q_{b1})q_{b1}+R^b-\tfrac{1}{2}K_b(s_{b0}-\mathbf{B})^2,$$

$$C^s = R^s - E_s[p_2(q_{a2}+q_{b2})]\mathbf{S}, \tag{3}$$

where E_i is the expectation operator for $i=\mathscr{A}, \mathscr{B}, \mathscr{S}$, and $K_a \geq 0$ and $K_b \geq 0$ are coefficients of constant absolute risk aversion, with one of K_a, K_b positive.[10] We have simplified notation by assuming the variance of each player's expectation is 1.[11] $\mathscr{S}$ is only active in the futures market; for $\mathscr{A}$ and $\mathscr{B}$, the payoffs relating only to the futures market are $R^a - E_a[p_2]\mathbf{A}$ and $R^b - E_b[p_2]\mathbf{B}$. Of course, $\mathbf{A}+\mathbf{B}+\mathbf{S}=0=R^a+R^b+R^s$. However, differences in expectations can make the futures market a positive-sum game.

4. Behaviour in the extraction subgame

Let net short positions $\mathbf{A}$ and $\mathbf{B}$ be given, positive or negative. The unique Cournot equilibrium which results in the extraction subgame is found by simultaneous maximization of C^a in q_{a1} and C^b in q_{b1} (with q_{i2} eliminated by (1)), subject to[12]

$$0 \leq q_{i1} \leq s_{i0}, \quad i=\mathscr{A}, \mathscr{B}. \tag{4}$$

The extraction equilibrium function $Q:\mathbf{R}^2 \mapsto [0, s_{a0}]\times[0, s_{b0}]$ is defined by

$$C^a(Q_a(\mathbf{A},\mathbf{B}), Q_b(\mathbf{A},\mathbf{B}), \mathbf{A}, R^a)$$
$$\geq C^a(z, Q_b(\mathbf{A},\mathbf{B}), \mathbf{A}, R^a) \quad \forall z \in [0, s_{a0}],$$

and

$$C^b(Q_b(\mathbf{A},\mathbf{B}), Q_a(\mathbf{A},\mathbf{B}), \mathbf{B}, R^b)$$
$$\geq C^b(z, Q_a(\mathbf{A},\mathbf{B}), \mathbf{B}, R^b) \quad \forall z \in [0, s_{b0}].$$

[10] This treatment of constant absolute risk aversion follows Newbury and Stiglitz (1981).

[11] In this model, no behavioural separation of risk aversion and variance in expectations is possible.

[12] The "path strategies" that arise in Reinganum and Stokey (1985) do not arise in subgame-perfect equilibrium here, unless stocks are made endogenous, or at least a third extraction period (post-maturity) is added. Each producer's marginal revenue is a declining function of the rival's extraction, which is sufficient for equilibrium to exist (Novshek, 1985).

When an interior maximum (w.r.t. (4)) occurs, the first-order condition is

$$\alpha - 2q_{i1} - q_{j1} = \alpha - 2q_{i2} - q_{j2} + \mathbf{I}$$
$$= \beta\eta_i \quad \text{for } \mathbf{I} = \mathbf{A}, \mathbf{B},\ i = \mathscr{A}, \mathscr{B},\ j = \{\mathscr{A}, \mathscr{B}\}\backslash\{i\}, \tag{5}$$

where the η_i's are Lagrangian Multipliers.

The solution to (5) is:[13]

$$q_{a1} = \tfrac{1}{2}s_{a0} + \tfrac{1}{6}\mathbf{B} - \tfrac{1}{3}\mathbf{A}, \qquad q_{a2} = \tfrac{1}{2}s_{a0} - \tfrac{1}{6}\mathbf{B} + \tfrac{1}{3}\mathbf{A},$$
$$q_{b1} = \tfrac{1}{2}s_{b0} + \tfrac{1}{6}\mathbf{A} - \tfrac{1}{3}\mathbf{B}, \qquad q_{b2} = \tfrac{1}{2}s_{b0} - \tfrac{1}{6}\mathbf{A} + \tfrac{1}{3}\mathbf{B}, \tag{6}$$

which sum to total extraction rates

$$q_1 = \tfrac{1}{2}(s_{a0} + s_{b0}) + \tfrac{1}{6}\mathbf{S}, \qquad q_2 = \tfrac{1}{2}(s_{a0} + s_{b0}) - \tfrac{1}{6}\mathbf{S}.$$

Each producer adjusts the amount of resource extracted prior to maturity for the futures market position he had taken, and partially counteracts the adjustment his rival makes. If $\mathbf{A} > 0$ and $\mathbf{B} < 2\mathbf{A}$, then $\mathscr{A}$ has taken a net short position, which he makes more profitable by shifting extraction from pre-maturity supply to at-maturity supply, driving down the value of the futures positions he has sold.

Equation (6) yields equilibrium cash market prices

$$p_1 = \hat{p} + \theta(\mathbf{A} + \mathbf{B}), \qquad p_2 = \hat{p} - \theta(\mathbf{A} + \mathbf{B}), \tag{7}$$

where $\theta = 1/(6\beta) > 0$, to simplify a recurrent term in expressions.[14] A higher θ indicates a steeper final demand for the resource. In a world of increasing spot prices — with a positive real interest rate — (7) means that a combined net short position of producers reduces the variability of spot prices.[15] Final demanders of the resource gain from this process, obtaining a larger gain (in present value) of consumer surplus in the cash market with a price below $\hat{p}$ than the loss in surplus suffered in the other cash market.

A critical aspect of the extraction equilibrium function Q is its independence from expectations[16] If a producer expected different intercepts α_t of

[13] Equations (6) characterize the extraction subgame equilibrium whenever they satisfy (4). Otherwise, the rival's net short position is replaced by that fraction of his net short position to which the rival responds, before reaching the constraint (4); in this case, one extraction rate depends upon both stocks.

[14] When the extraction equilibrium is not interior to (4), still $\mathbf{S} < 0 \Leftrightarrow p_1 > \hat{p}$.

[15] However, comparison with a monopolist with initial stock $s_{a0} + s_{b0}$ and net short position $\mathbf{A} + \mathbf{B}$ would further reduce spot price variability: the coefficient θ in (7) would be replaced by 1.5θ (Brianza, Phlips and Richard, 1987).

[16] This independence allows applying the subgame-perfect equilibrium concept to a game of inconsistent incomplete information. For further discussion, see Phlips and Harstad (1990).

the final demand curve in cash markets at different dates t, the time path of extraction would be affected; but the extent to which extraction is shifted into the period in which futures contracts mature, from pre-maturity cash markets, does not depend upon the producer's beliefs about the general level of final resource demand. In particular, this separates futures game behaviour from extraction subgame behaviour to the extent that a trading partner of a producer on the futures market need not know the producer's expectations in order to predict the extent to which he will shift extraction in reaction to a contract signed.[17]

5. Accounting for equilibrium in the subgame

The game tree is common knowledge; all players can calculate the function Q that will characterize behaviour in the extraction subgame for any $\mathbf{A}$, $\mathbf{B}$ determined in the futures market. Let the mean expectations of $\hat{p}$ for $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ be $\hat{p}_a$, $\hat{p}_b$, $\hat{p}_s$.[18] Focusing on subgame-perfect equilibria by incorporating (6) into the payoffs yields:[19]

$$\begin{aligned}\pi^a &= R^a + \hat{p}_a(s_{a0} - \mathbf{A}) + \tfrac{1}{3}\theta\mathbf{S}^2 - \tfrac{1}{2}K_a(s_{a0} - \mathbf{A})^2,\\ \pi^b &= R^b + \hat{p}_b(s_{b0} - \mathbf{B}) + \tfrac{1}{3}\theta\mathbf{S}^2 - \tfrac{1}{2}K_b(s_{b0} - \mathbf{B})^2, \qquad (8)\\ \pi^s &= R^s - \hat{p}_s\mathbf{S} - \theta\mathbf{S}^2.\end{aligned}$$

Comparing (8) and (3) indicates that a speculator trading with a producer, should he fail to take into account the producer's market power in the cash market, would be neglecting a quadratic term with negative coefficient in

[17] It is important, however, for a trading partner to know how far from binding constraint (4) is.

[18] Note that differences in beliefs are not the result of differences in information; no player updates his beliefs upon learning that rivals have different beliefs. Thus, these are pure differences in *opinion* in the terminology of Varian (1989). Notice also that inconsistent prior beliefs avoid the impossibility theorems for speculative trading in rational expectations equilibrium (e.g., Kreps, 1977; Tirole, 1982; Milgrom and Stokey, 1982). Null hypotheses that give rational expectations wide latitude nonetheless fail to organize data on prices in metals futures markets (cf., MacDonald and Taylor, 1988; Hall and Taylor, 1988, and references therein).

[19] The model has multi-stage Nash equilibria which we ignore because they fail to be subgame-perfect. To wit, they involve threats by a producer to punish deviations from the equilibrium path in the futures market via altering his extraction policy to harm both himself and the player being threatened. Since the futures market is closed before the producers can commit themselves to extraction plans, such threats are not credible, and such equilibria are not sensible.

an otherwise linear payoff function![20] The producer need not have any market power in the futures market per se for this concern to be serious.

A producer choosing to be inactive in the futures market nonetheless profits from its existence. That is, if $\mathbb{A} = 0 \neq \mathbb{B}$, then $\mathbb{S} = -\mathbb{B}$, and $\mathscr{A}$ attains a benefit in the quadratic term in (8). This benefit results from $\mathscr{B}$ shifting extraction from a higher-priced to a lower-priced period (so $\mathscr{B}$ can enhance profitability of his futures contracts), and $\mathscr{A}$ having a profitable opportunity to counteract this shift.

Finally, the futures game specified by payoffs (8) will yield purely speculative trades. To be precise, the alterations to (8) that would result from trivializing the extraction subgame by setting $s_{a0} = s_{b0} = 0$ would still yield a positive-sum game, whenever futures positions were sufficiently small relative to differences in beliefs.[21]

6. Futures market behaviour

Subgame-perfect equilibrium behaviour involves best responses at every node of the game tree, so best responses must be specified backwards from the end of the game. Step 6 was treated in Section 4 abo͂ve. To handle acceptance behaviour, let the set of permutations of the ordered triple $(\mathscr{A}, \mathscr{B}, \mathscr{S})$ be Ω, and an arbitrary element be (3, 4, 5), and designate vectors of offered prices $P := (p^a, p^a, p^b, p^b, p^s, p^s)$ and maximum quantities $F := (f^a, f^a, f^b, f^b, f^s, f^s)$. Each π^i, $i = \mathscr{A}, \mathscr{B}, \mathscr{S}$, is treated as a real-valued function with arguments: P, the four acceptances by i, and the eight other acceptances. The constraint that an acceptance cannot be more than the currently available amount is specified by feasible sets:

$$\Delta_5 := [0, f^3 - f_4^3] \times [0, f^3 - f_4^3] \times [0, f^4 - f_3^4] \times [0, f^4 - f_3^4],$$

$$\Delta_4 := [0, f^3] \times [0, f^3] \times [0, f^5 - f_3^5] \times [0, f^5 - f_3^5],$$

$$\Delta_3 := [0, f^4] \times [0, f^4] \times [0, f^5] \times [0, f^5].$$

To characterize Step 5 behaviour, let $v_5 := (f_5^3, f_5^3, f_5^4, f_5^4)$, the vector of acceptances. The history of acceptances up to Step 5 is

$$h_5 := (f_3^4, f_3^4, f_3^5, f_3^5, f_4^3, f_4^3, f_4^5, f_4^5).$$

[20] In the much lengthier payoff functions when (4) is binding, the quadratic term involving $\mathbb{S}$ stops increasing, as speculator-producer trading continues, at the level where (4) became binding. If the producer is risk-averse, however, equilibrium positions will still be finite. One producer being risk-averse makes gains from producer-producer trades quadratic, hence these positions are also finite in equilibrium.

[21] For finite trading in equilibrium, there would have to be at most one risk-neutral player.

The best response function $V_5:\mathbf{R}_+^{14}\mapsto\Delta_5$ is defined by

$$\pi^5(P,\ V_5(P,h_5),\ h_5)\geq\pi^5(P,\ v_5,\ h_5)\quad\forall v_5\in\Delta_5. \tag{9}$$

V_5 is the restriction onto Δ_5 of a linear function. When the offer most profitable to 5 is (p^3, f^3), V_5 takes the form $Y+Z^{-1}p^3$, with Y reflecting $\hat{p}_5$ and 5's net short position on trades already completed, Z reflecting θ and K_i.[22]

To characterize Step 4 behaviour, let $u_4:=(f_4^3, f_4^3, f_4^5, f_4^5)$, the vector of acceptances. The history of acceptances up to Step 4 is

$$g_4:=(f_3^4, f_3^4, f_3^5, f_3^5).$$

The best response function $U_4:\mathbf{R}_+^{10}\mapsto\Delta_4$ is defined by

$$\begin{aligned}&\pi^4(P,\ U_4(P,g_4),\ g_4,\ V_5[P,\ g_4,\ U_4(P,g_4)])\\&\quad\geq\pi^4[P,\ u_4,\ g_4,\ V_5(P,\ g_4,\ u_4)]\quad\forall u_4\in\Delta_4.\end{aligned} \tag{10}$$

So player 4 incorporates the value of 5's best response function, for each decision 4 might make, in the right-hand side of (10). The left-hand side of (10) is his equilibrium payoff, given P, g_4. U_4 is a complicated, piecewise linear function, with slopes and intercepts similar to V_5 in cases when V_5 does not react locally to U_4, and marginal insensitivity to prices in some cases where the partial derivative of V_5 with respect to u_4 is 1 or -1.

To characterize Step 3 behaviour, let $t_3:=(f_3^3, f_3^3, f_3^5, f_3^5)$, the vector of acceptances. The best response function $T_3:\mathbf{R}_+^6\mapsto\Delta_3$ is defined by

$$\begin{aligned}&\pi^3[P,\ T_3(P),\ U_4[P,\ T_3(P)],\ V_5(P,\ T_3(P),\ U_4[P,\ T_3(P)])]\\&\quad\geq\pi^3(P,\ t_3,\ U_4(P,t_3),\ V_5[P,\ t_3,\ U_4(P,t_3)])\quad\forall t_3\in\Delta_3.\end{aligned} \tag{11}$$

So player 3 incorporates the value of 4's best response function, for each decision 3 might make, and the value of 5's best response to 4's best response, for each decision 3 might make, in the right-hand side of (11). The left-hand side of (11) is his equilibrium payoff, given P. T_3 is a quite complicated, piecewise linear function.

In Step 2, each permutation (i,j,k) in $\Omega=\{(\mathcal{A},\mathcal{B},\mathcal{S}),\ldots,(\mathcal{S},\mathcal{B},\mathcal{A})\}$ occurs with probability $\frac{1}{6}$. Thus, the expected payoff given (P,Q), for player

[22] When $5=\mathcal{S}$, $Z=2\theta$; if 5 is a producer accepting the other producer's offer, $Z=K_5$; otherwise, $Z=K_5-\frac{3}{2}\theta$, if this is positive (if not, an interior acceptance does not occur). If $f_5^3=f^3-f_4^3$, this is now figured into the intercept term for f_5^4's linear equation. The sign of Z is reversed throughout if 5 chooses to accept offers to sell (in equilibrium, he does not accept both offers of one other player during Step 5).

$l = \mathscr{A}, \mathscr{B}, \mathscr{S}$, is a function $G^l: \mathbf{R}_+^{12} \to \mathbf{R}$ defined by

$$G^l(P, Q) := \frac{1}{6} \sum_{(i,j,k) \in \Omega} \pi^l[P, T_i(P), U_j[P, T_i(P)],$$
$$V_k(P, T_i(P), U_j[P, T_i(P)])].$$

This definition has best response behaviour at each step incorporated as history in the following steps.[23]

During Step 1, offers

$$(p^a, p^a, p^b, p^b, p^s, p^s, f^a, f^a, f^b, f^b, f^s, f^s)$$

satisfy subgame perfection if:

$$\begin{aligned}
&G^a(p^a, p^a, p^b, p^b, p^s, p^s, f^a, f^a, f^b, f^b, f^s, f^s)\\
&\quad \geq G^a(p, p, p^b, p^b, p^s, p^s, f, f, f^b, f^b, f^s, f^s)\\
&\quad\quad \forall (p, p, f, f) \geq 0,\\
&G^b(p^a, p^a, p^b, p^b, p^s, p^s, f^a, f^a, f^b, f^b, f^s, f^s)\\
&\quad \geq G^b(p^a, p^a, p, p, p^s, p^s, f^a, f^a, f, f, f^s, f^s)\\
&\quad\quad \forall (p, p, f, f) \geq 0,\\
&G^s(p^a, p^a, p^b, p^b, p^s, p^s, f^a, f^a, f^b, f^b, f^s, f^s)\\
&\quad \geq G^s(p^a, p^a, p^b, p^b, p, p, f^a, f^a, f^b, f^b, f, f)\\
&\quad\quad \forall (p, p, f, f) \geq 0.
\end{aligned} \tag{12}$$

The next section discusses implications of (12).

7. Subgame-perfect offers

All subgame-perfect contracts have the offeror fixing the price, and later the acceptor fixing the quantity. The basic considerations involved in payoff-maximizing offers can be illustrated graphically. Figure 1 will aid in seeing how prices to be announced are determined by anticipating the quantities that would be accepted for different prices which could be announced, and in seeing what calculations guide the choice of trading partner to focus on in determining an announcement. In subgame-perfect equilibrium, changing the maximum quantity announced will affect quantities accepted only if the constraint that no more can be accepted than is available should happen to be binding. Thus, Figure 1 will only indicate lower bounds for the maximum quantity that should be announced.

[23] F affects G^h via restrictions on the range of T_i, U_j and V_k.

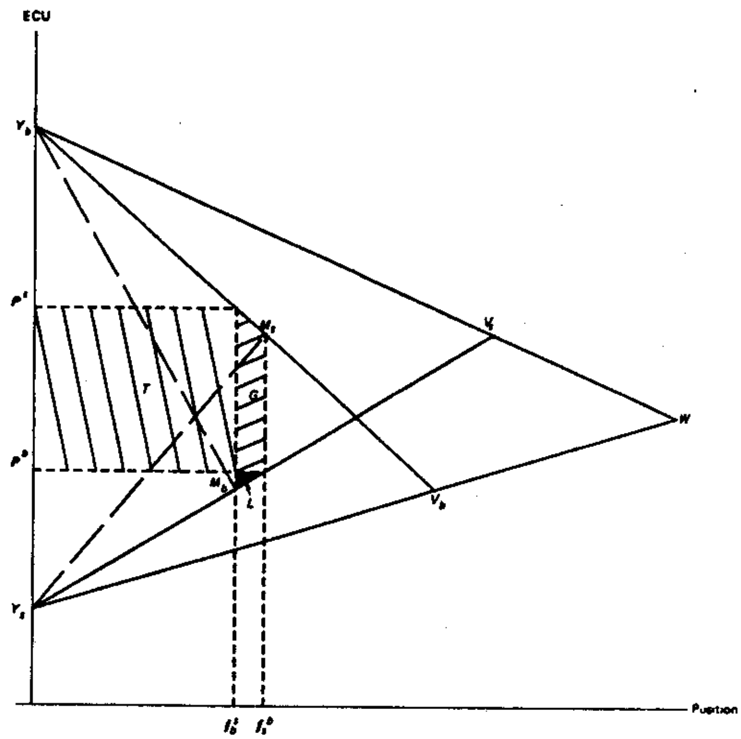

Figure 1.

To reach a relatively simple diagrammatic analysis, initially one of the players is ignored. For convenience, focus on the two contracts f_b^s and f_s^b where $\mathcal{S}$ sells to $\mathcal{B}$, assuming no gains from trade could result from $\mathcal{B}$ selling to $\mathcal{S}$. Naturally, the price p^s at which $\mathcal{S}$ announces he would like to sell will exceed the price p^b at which $\mathcal{B}$ announces he would like to buy.

Because each prefers his own announced price, in commitment orders where $\mathcal{S}$ accepts before $\mathcal{B}$, $\mathcal{S}$ will set $f_s^b = 0$. Similarly, in the remaining commitment orders, $\mathcal{B}$ will set $f_b^s = 0$, knowing that $\mathcal{S}$ in a later step will accept $\mathcal{B}$'s price p^b rather than forego all gains from trade with $\mathcal{B}$. (We continue to ignore $\mathcal{A}$ for the moment.)

With the quantity of $\mathcal{S}$'s futures sales to $\mathcal{B}$ on the horizontal axis in Figure 1, and the price shown vertically, we can create an analogue to an Edgeworth box diagram: any (quantity, price) vector inside the "lens" Y_sY_bW represents a mutually beneficial trade. Differences in beliefs, size of hedgable stocks, and net positions on contracts with $\mathscr{A}$ determine the intercepts Y_s and Y_b. When $\mathscr{B}$ is determining the quantity f_b^s after $f_s^b = 0$, $\mathscr{B}$ maximizes his payoff by setting f_b^s for any p^s halfway across from 0 to $Y_b W$, that is, along the line $Y_b V_b$. Similarly, if f_b^s has been set at zero, $\mathscr{S}$ will choose to respond to p^b by setting f_s^b along $Y_s V_s$, which bisects the angle $Y_b Y_s W$.

Note. Let $(\mathbb{S}\backslash f_b^s)$ denote $\mathscr{S}$'s net short position under the assumption $f_b^s = 0$, with $(\mathbb{S}\backslash f_s^b)$, $(\mathbb{B}\backslash f_b^s)$, $(\mathbb{B}\backslash f_b^a)$ analogous incomplete net short positions. Assuming the only contract between $\mathscr{B}$ and $\mathscr{S}$ is f_s^b, $Y_s = -(\mathbb{S}\backslash f_s^b) - \hat{p}_s/2\theta$, which may be positive or negative (the horizontal axis need not imply a zero price, though the lens is bounded below by zero prices and quantities). The slope of $Y_s W$ is $(4\theta)^{-1}$; this and the intercept are obtained by solving $\pi_s - \pi_s|_{f_s^b = f_b^s = 0} = 0$.

Assuming the only contract between $\mathscr{B}$ and $\mathscr{S}$ is f_b^s,

$$Y_b = (\hat{p}_b + \tfrac{2}{3}\theta(\mathbb{S}\backslash f_b^s) + K_b[s_{b0} - (\mathbb{B}\backslash f_b^s)])/(K_b - \tfrac{2}{3}\theta),$$

assuming the denominator is positive (otherwise, f_b^s will be either 0 or $f^s - f_a^s$). The slope of $Y_b W$ is $-(2K_b - \tfrac{4}{3}\theta)^{-1}$; this and the intercept are obtained by solving $\pi_b - \pi_b|_{f_s^b = f_b^s = 0} = 0$. If, instead, we were focusing on f_b^a and f_b^a, Y_b would be

$$\hat{p}_b / K_b - (s_{b0} - (\mathbb{B}\backslash f_b^a)),$$

and $Y_b W$ would have slope $(2K_b)^{-1}$. Analogous results would be obtained for $\mathscr{A}$.

Now consider what price $\mathscr{S}$ should announce. The price he sets will be relevent when he will get to set $f_s^b = 0$ before $\mathscr{B}$ determines f_b^s. Assuming this, he should act like a monopolist. That is, he draws a "marginal revenue" line $Y_b M_b$, and sets price p^s so that $\mathscr{B}$ will later choose to set f_b^s equal to the quantity where $Y_b M_b$ reaches $Y_s V_s$ ($\mathscr{S}$'s "marginal cost"), as shown.

In the reverse situation, $\mathscr{B}$ should act like a monopsonist, drawing line $Y_s M_b$, and setting price p^b so that $\mathscr{S}$ will later choose to set f_s^b equal to the quantity where $Y_s M_s$ reaches $Y_b V_b$, as shown.

How large is the payoff difference to $\mathscr{S}$ that results from trading with $\mathscr{B}$ on his own terms rather than (p^b, f_s^b)? This payoff difference is the area of the rectangle T, the transfer amount, less the area of the triangle L, $\mathscr{S}$'s share of the efficiency loss between f_b^s and f_s^b. The payoff difference to $\mathscr{B}$ between his own terms and $\mathscr{S}$'s terms is the sum of the areas of the rectangle T and the trapezoid G ($\mathscr{B}$'s share of the efficiency gain). Thus, the player with the steeper V_i function (cf. (9)) will have more at stake in the determination of terms of trade.

While a number of complicated issues arise, principally with determining the intercepts for diagrams like Figure 1, the central theme of the Step 1 decisions is a comparison of the amounts at stake for each contract. That is, $\mathcal{S}$ determines (p^s, f^s) essentially by calculating whether more payoff difference is at stake in (p^s, f^s_b) versus (p^b, f^b_s) than in (p^s, f^s_a) versus (p^a, f^a_s), each weighted by the number of commitment orders for which he could dictate his terms to the rival in question.

8. Efficiency

Proposition. *Given divergent prior beliefs, in subgame-perfect equilibrium, due to the rule which has one party to a contract fixing the price and the other determining the quantity, there remain volume-enhancing, mutually beneficial trades of futures contracts.*

Remark. The proof is straightforward; Pareto-efficiency so constrains contracts as to give some party a profitable opportunity to have offered to buy at a lower, or sell at a higher price, reducing volume in the process.

Proof of the proposition. Suppose, contrary to the proposition, that there exists a set of parameters with $\hat{p}_a$, $\hat{p}_b$, $\hat{p}_s$ all distinct, and an associated subgame-perfect equilibrium behaviour which yields a Pareto-efficient outcome, in that no mutually beneficial trades remain. Hence, no mutually beneficial trades may remain for the pattern of acceptances occurring in any commitment order. Thus, in an arbitrary commitment order $\omega \in \Omega$, for each pair of players $i, j \in \{\mathcal{A}, \mathcal{B}, \mathcal{S}\}$, the following property must be satisfied (perhaps by relabelling): (†): {j has accepted a contract offered by i, at precisely the same net position that i would have chosen at that price, had i been the one accepting}. Under these circumstances, it would be inconsistent with equilibrium for i to prefer to trade with j at the terms j offered rather than at i's own terms. So j offered either identical or less favourable terms to i. Across commitment orders, there are 18 pairs constraining an offer by one of the pair to satisfy (†). As only 6 offers are made, either (i) every offer must be accepted and satisfy (†) in at least 3 commitment orders, or (ii) some offer (††) must be accepted and satisfy (†) in at least 4 commitment orders. Option (i) constrains some i, j pair to offer each other identical prices and maximum quantities sufficient to reach a Pareto-efficient outcome; so i can gain by announcing the same maximum quantity at a price slightly more favourable to him, as in the commitment orders where i follows j, he cannot be worse off, since j is by presumption offering him his original terms. Thus, option (i) yields a contradiction to the presumed equilibrium, in the direction of reduced volume of futures contracts for those commitment orders where j accepts i's terms. Option (ii) is similarly inconsistent with equilibrium, as the player i offering (††) has available to him sufficiently favourable terms in the remaining two commitment orders to be able to alter (††) backward along the steepest V_j accepting it, attaining a higher payoff. □

If our stylized version of trading rules on futures markets is capturing some essentials of rules and incentives in actual futures markets, such as the London Metal Exchange, then the proposition calls for a more careful look at the incentives facing the market-makers who determine which rules are to be employed. Market-makers have not been introduced explicitly in this model, nor have commissions. Both sides to each trade in subgame-perfect equilibrium, however, would be willing to pay some positive commission to a market-maker and still accept the trades consummated.[24] A market-maker looking to maximize commissions via maximizing total futures contract volume would be expected to investigate whether an alternative set of trading rules could be found which would attain the higher Pareto-efficient trading volume. In fact, such a set of rules may exist: Phlips and Harstad (1990) find that subgame-perfect equilibria reach Pareto-efficient volume when any trader accepting a contract must accept the full maximum quantity offered, that is, when a potential acceptor must simply say "yes" or "no", not "I'll take 100 of that". (All other rules unchanged.)

If such "all-or-nothing" rules were used, in subgame-perfect equilibrium any contract acceptor obtains just enough profit to induce him to agree; it is the appropriation of all gains from trade (in excess of this minimum) that leads the contract's offeror to announce a Pareto-efficient position. If commissions were introduced, they would figure into the minimum profit level allowed an acceptor. Averaging across commitment orders, all three players would expect positive profits, but every contract that generates a commission for the market-maker would require the signature of a party nearly indifferent to signing. This uneven distribution of profit, on a contract-by-contract basis, may likely impose some notable cost or disutility upon

[24] A nonzero marginal commission would, naturally, reduce total volume of contracts, but may not affect equilibrium net short positions.

the market-maker, which would explain why market-makers appear to prefer using rules more like those we have studied in this paper, even at the expense of a reduced commissionable trading volume. If the model were extended to allow for some uncertainty about trading partners' prior beliefs, it is possible that expected trading volume would be higher when trades by design have not-too-uneven profit splits (because the acceptor determines the quantity traded), than when the offeror attempts a total profit appropriation. Extending the model in this direction would take us hopelessly far afield.

9. Concluding remarks on uniform prices

It is customary in models of futures markets to impose a uniform price upon all trades in one market period of futures contracts with a common maturity date. Most models then proceed to ask whether this one price is an unbiased predictor of the spot price at maturity.

When differences in opinion are the source of gains from purely speculative trade on futures markets, market participants are not looking to "the" futures price to aggregate or provide information about the spot price at maturity; they are not about to modify their beliefs upon learning that "the market" has different beliefs.

Only with some violence can the contract data observed on futures markets be fitted into a uniform price model. Although simple in many respects, our model nonetheless produces a diversity of prices, which seems entirely natural. It is even possible to find parameters for which a subgame-perfect equilibrium outcome involves multiple prices on contracts between the same trading pair.

Unlike capital asset pricing models currently in vogue, no efficient markets assumption is employed, and equilibrium outcomes are inefficient. The root of suboptimal volume of futures activity is not the small number of participants, but must be traced to the procedure wherein a trader can accept a part of the quantity offered for sale or purchase, at the offered price, without having to accept the entire offer. However, only an oligopolistic and game-theoretic model can allow analysis of such impacts of trading rules upon outcomes.

Finally, in a very real sense, a uniform price on a futures market would itself be a source of inefficiency. Whatever the volume of trade that occurs at a uniform price, there remain gains from further trade, for almost any pair of traders, which can be reached only by discarding the uniform price.

Except for pathological parameter sets, subgame-perfect equilibrium outcomes in this model involve at least one contract between each pair of traders, such that expected profit to each trader on that contract (given inconsistent prior beliefs) is bounded above zero. Thus, at least one trader will be both buying and selling, viewing both as profitable. This realistic feature is only possible when a uniform price rule is discarded.

References

Anderson, R.W. (1984) *The Industrial Organization of Futures Markets*. Lexington, MA: Lexington Books.

Brianza, T., L. Phlips and J.-F. Richard (1987) 'Futures markets, speculation and monopoly', CORE Discussion Paper 8725, Louvain-la-Neuve.

Damme, E. van (1987) *Stability and Perfection of Nash Equilibria*. Berlin: Springer.

Hall, S.G. and M.P. Taylor (1988) 'Modelling risk premia in commodity forward prices: some evidence from the London metal exchange', Economics Division Discussion Paper, Bank of England, London.

Harsanyi, J.C. and R. Selten (1988) *A General Theory of Equilibrium Selection in Games*. Cambridge, MA: M.I.T. Press.

Kreps, D. (1977) 'A note on 'fulfilled expectations' equilibria'', *Journal of Economic Theory*, 14:32-43.

MacDonald, R. and M.P. Taylor (1988) 'Testing rational expectations and efficiency in the London metal exchange', *Oxford Bulletin of Economics and Statistics*, 50:41-52.

Milgrom, P.R. and N.L. Stokey (1982) 'Information, trade and common knowledge', *Journal of Economic Theory*, 26:17-27.

Newbury, D.M. and J.E. Stiglitz (1981) *The Theory of Commodity Price Stabilisation: A Study in the Economics of Risk*. Oxford: Clarendon Press.

Novshek, W. (1985) 'On the existence of Cournot equilibrium', *Review of Economic Studies*, 52:85-98.

Phlips, L. and R.M. Harstad (1990) 'Interaction between resource extraction and futures markets: a game-theoretic analysis', in: R. Selten, ed., *Game Equilibrium Models*. Heidelberg: Springer.

Reinganum, J.F. and N.L. Stokey (1985) 'Oligopoly extraction of a common property natural resource: the importance of the period of commitment in dynamic games', *International Economic Review*, 26:161-173.

Selten, R. (1982) 'Einführung in die Theorie der Spiele mit unvollständiger Information', in: *Information in der Wirtschaft, Schriften des Vereins für Sozialpolitik, Gesellschaft für Wirtschafts- und Sozialwissenschaften (N.F. Band 126)*. Berlin: Duncker & Humbolt, pp. 81-147.

Tirole, J. (1982) 'On the possibility of speculation under rational expectations', *Econometrica*, 24:74-81.

Varian, H.R. (1989) 'Differences in opinion in financial markets', in: C. Stone, ed., *Financial Risk: Theory, Evidence and Implications*. Amsterdam: Kluwer Academic Publishers.