LOTTERY QUALIFICATION AUCTIONS

Ronald M. Harstad and Robert F. Bordley

ABSTRACT

We analyze a Q-lottery qualification auction: the Q highest bidders qualify for a lottery giving each a 1/Q chance of obtaining the asset (Q = 1 is a second-price auction). We also provide some results for an oral variant, the Q-curtailed oral auction, which sets the price as soon as only Q of n bidders remain in competition, then awards the asset by lottery. Despite the probability of an inefficient outcome, there are many cases in which a seller prefers to choose Q > 1. Examples show that Milgrom and Weber's Linkage Principle does not extend to nonstandard auctions. In particular, undermining the privacy of the highest-valuing bidder's information and augmenting expected revenue are seen to be less closely aligned than previous explanations might suggest.

I. INTRODUCTION

In a "lottery qualification" auction, bids are submitted for the purpose of qualifying for a lottery to determine the winner. Specifically, a seller announces a parameter Q, bidders submit sealed bids, and the Q highest bidders qualify for a lottery in which each has a 1/Q chance of being the
winning bidder. The lottery winner obtains the asset at a price equal to the highest bid rejected for qualification: the $(Q + 1)$st highest bid. The standard second-price auction corresponds to $Q = 1$.

As this paper shows, when the value of the auctioned asset involves uncertainty common to all bidders (e.g., resale value), setting $Q > 1$ leads rational bidders to bid more aggressively (since winner’s curse corrections are lessened). As a result, equilibrium expected revenue for lottery qualification auctions can exceed that for the standard auction forms considered in Milgrom and Weber (1982).

Suppose an auctioned asset has a common value $\nu$ to any of three bidders, with $\nu$ distributed uniformly on a wide range $[L, H]$. Let each bidder $i$ observe an estimate $X_i$, with a uniform conditional distribution on $[\nu - 1, \nu + 1]$, given that $\nu = \nu_i$. In such a setting, a first-price (sealed-bid) auction would have a symmetric equilibrium bid function approximated by $b_i(x) = x - 1$, with expected revenue about $\overline{V} - \frac{1}{2}$, where $\overline{V} = (L + H)/2$. A second-price (sealed-bid) auction would have a symmetric equilibrium bid function approximated by $b_i(x) = x - \frac{1}{2}$, with expected revenue about $\overline{V} - \frac{1}{2}$. An English auction would have the first bidder quit competing at $b_0(x) = x$, and the remaining bidders quit at $b_i(x, p_0) = (x + p_0)/2$, where $p_0$ is the price at which the first bidder quit, with equilibrium expected revenue about $\overline{V} - \frac{1}{4}$. All these revenue comparisons follow the general pattern provided in Milgrom and Weber.

The $Q = 2$ lottery qualification auction outperforms all three standard auctions in this example. If bidders know that the lowest bidder sets the price, and the two higher bidders each have a $\frac{1}{2}$ chance of obtaining the asset at that price, the symmetric equilibrium bid function is approximated by $b_i(x) = x + \frac{1}{2}$, with expected revenue about $\overline{V} - \frac{1}{2}$. In essence, switching from a second-price auction (which is a $Q = 1$ lottery qualification auction) to $Q = 2$ changes the price from the median to the lowest bid, but leads bidders rationally to increase their bids by more than the expected distance between the median and the lowest signal.

Setting $Q > 1$ gains because a bidder rationally responds to a higher $Q$ by bidding with sufficiently increased aggressiveness so as to enhance expected revenue. This only occurs when asset value has a strong common-value element (as, for example, when later resale is feasible). Thus, in pure private-values models in which a bidder’s asset value is independent of other bidders’ information (an extreme assumption), each bidder would bid his expected asset value for any $Q$, and the seller rationally sets $Q = 1$. 

Observations of laboratory auctions suggest caution in advising sellers to adopt these techniques, as even highly experienced bidders appear to have substantial difficulties in learning rational responses to auction incentives (as opposed to adequate rules of thumb for particular auction situations, cf. Kagel, Levin, and Harstad, 1995; Dyer, Kagel, and Levin, 1989). Our near-term interest in these auction techniques is more elucidative, however, analysis of lottery-qualification auctions offers a further insight into revenue comparison logic. Section IV outlines this intuition, focusing on the disparity between the price-setting bid and the estimate of asset value that the price setter would calculate, given his private information and being informed that he had set the price. Reducing this disparity increases expected revenue.

II. EQUILIBRIA IN THE GENERAL SYMMETRIC MODEL

This section presents definitions and characterizations of symmetric equilibrium bidding in Q-lottery qualification auctions and Q-curtailed oral auctions. The auctions are set in the environment introduced in Milgrom and Weber (1982) as the General Symmetric Model; the special case of a common-value environment is considered in some examples that follow. A discussion of the model, its assumptions and relation to the literature is presented in pages 1090–1100 of their paper; only key provisions are mentioned in this section.

An indivisible asset is sold by auction. Each bidder \( i \in \mathbb{N} = \{1, \ldots, n\} \) possesses private information represented by a real-valued signal \( X_i \), with \( X = (X_1, \ldots, X_n) \). In addition, \( S = \{S_1, \ldots, S_m\} \) represent a set of nonparticipant appraisals. The asset’s value to bidder \( i, V_i \), may depend on variables unknown to him.

**Assumption 1.** The joint distribution of the random variables \((S, X)\) is affiliated and is exchangeable in the \( X_i \)'s.

**Assumption 2.** There are nondecreasing continuous functions \( \zeta_j, j \in \mathbb{N}, \) so that \( V_i = \zeta(X, S) = \zeta(X_i, (X_j)_{j \neq i}, S) \). That is, each bidder’s value is a symmetric function of rivals’ information, and depends on nonparticipant appraisals in the same way.

Seller and all bidders are assumed risk-neutral. When considering symmetric behavior, we focus on bidder 1, and let \((Y_1, \ldots, Y_{n-1})\) be \((X_2, \ldots, X_n)\) arrayed descendingly, that is, the order statistics of bidder 1’s rivals. It is also convenient to have notation for all \((X_1, \ldots, X_n)\).
arrayed descendingly (all the order statistics), let this be \((Z_1, \ldots, Z_n)\). Thus, \(Y_1\) is the highest signal observed by bidder 1's rivals, \(Y_{Q,i}\) is the signal observed by the rival whom bidder 1 must outbid in order to qualify, and \(Z_{Q+1}\) is the signal observed by the bidder who will end up setting the price. Let \(X\) denote the support of \(X_i\). For simplicity, we assume throughout that the event \(\{X_1 = Y_1\}\) has zero probability.

Deriving the symmetric equilibrium bid in the \(Q\)-lottery-qualification auction is a straightforward generalization of the derivation for the second-price auction \((Q = 1)\). For \(1 \leq Q \leq n - 1\), define

\[
v_Q(x, y) := E[V_1 \mid X_1 = x, Y_Q = y].
\]

(1)

Affiliation implies that \(v_Q\) is nondecreasing; following Milgrom and Weber, we add the presumably harmless assumption that it is increasing in its first argument. Let \(b_Q(x) := v_Q(x, x)\). It is straightforward to show that \(b_Q(x) \geq b_{Q-1}(x)\), with equality degenerate.

**Proposition 1.** The strategy profile \((b_0, \ldots, b_Q)\) is the unique symmetric equilibrium for the \(Q\)-lottery qualification auction.

The proof, which closely follows Theorem 6 in Milgrom and Weber, is in the Appendix.

Notice that in equilibrium, the \(Q\) bidders with the \(Q\) highest signals qualify for the lottery. To the extent that interpersonal differences in asset value prevail, there is only a \(1/Q\) chance of an allocatively efficient outcome (the chance that the highest bidder wins the lottery). The intuition as to why \(b_Q\) is a symmetric equilibrium bid function is discussed in Section IV below.

We use the term "curtailed oral" auction for an oral version of the auction-cum-lottery: the price rises continuously until exactly \(Q\) bidders remain in competition. The price is thereby set (further competition is curtailed), with the \(Q\) remaining bidders qualifying for the lottery giving each of them a \(1/Q\) chance of obtaining the asset at the set price (thus, \(Q = 1\) is the special case of an English auction).

Just as the notation needed to specify equilibrium for the English auction is more complex than that for the second-price auction (cf. Milgrom and Weber, 1982), so too must added notation be introduced for the \(Q\)-curtailed oral auction. A strategy for bidder \(i = 1, \ldots, n\) is a vector of functions \(B_i := (B_{i0}, B_{i1}, \ldots, B_{i(n-Q+1)})\), where \(B_{ik}(x, p_1, \ldots, p_k) \geq p_i\) is bidder \(i\)'s planned quit price as a function of his signal realization \(x_i\), given that \((p_1, \ldots, p_k)\) are the prices at which the first \(k\) rivals quit.
and assuming no other rival quits before \( B_k \) (otherwise, the relevant \( M_{i-1} \) is used instead). Consider the strategy \( B^Q = (B_0^Q, B_1^Q, \ldots, B_{n-Q-1}^Q) \), where

\[
B_0^Q(x) = E[V_1 | X_1 = Y_Q = \cdots = Y_{n-1} = x],
\]

\[
B_k^Q(p_1, \ldots, p_k) = E[V_1 | X_1 = Y_Q = \cdots = Y_{n-k-1} = x],
\]

\[
B_k^Q(Y_{n-k}, p_1, \ldots, p_{k-1}) = p_k, \ldots, B_0^Q(Y_{n-1}) = p_1.
\]

(2)

**Proposition 2.** The strategy profile \((B^Q, \ldots, B^Q)\) is a symmetric equilibrium of the \(Q\)-curtailed oral auction.

The proof, which closely follows Theorem 10 in Milgrom and Weber, is in the Appendix.

**Proposition 3.** For \( Q \leq n - 1 \), expected revenue in the \(Q\)-curtailed oral auction is at least as large as in the \(Q\)-lottery-qualification auction. Expected revenue in either the \(Q\)-lottery-qualification auction or the \(Q\)-curtailed oral auction is enhanced by public announcement of any information seller possesses which is affiliated with asset value.

Proofs follow Milgrom and Weber’s results with modifications as in the Appendix; details are omitted. Note that for \( Q = n - 1 \), the two actions are strategically equivalent, as the first bidder to quit in the curtailed oral auction sets the price, so the price cannot be informed by inferring the private information of rivals who quit earlier.

### III. EXPECTED REVENUE COMPARISONS

This section presents revenue consequences of equations (1) and (2), assuming that each bidder’s valuation of an asset is given by

\[
V_i = \lambda \Omega + (1 - \lambda)X_i,
\]

(3)

where \( \Omega \) is a common random variable, perhaps reflecting resale value, and \( \lambda \) represents the degree to which the auction is common-value (\( \lambda = 1 \)) or private-values (\( \lambda = 0 \)). \( Q = 1 \) will always be revenue-maximal when \( \lambda \) is far from 1 (i.e., when the auction is primarily private-values).
A general theorem comparing expected revenue as $Q$ varies seems to be unattainable, as comparison of expectations of different conditions expectations evaluated at different order statistics quickly becomes impenetrable. We examine four particular cases:

1. $\Omega = (\frac{1}{n})\sum X_i$, with the $X_i$ i.i.d. uniform. This example is pathological, in that the information linked to price by increasing $Q$ is not correlated with the winning bidder's private information. Thus, the Linkage Principle would not predict revenue increases with $Q$; indeed, revenue falls.

2. $\Omega$ is uniform, the $X_i$ are uniform and unbiased given $\Omega$, so asset values are affiliated. In this case, expected revenue strictly increases as $Q$ increases.

3. $\lambda = 1$, $\Omega$ is uniform and $X_i's$ conditional distribution is triangular, i.e., asset values are affiliated and signals are conditionally non-uniform, with more accurate signals more likely. In this case, expected revenue increases until $Q \approx n/3$, and then falls as $Q$ increases further.

4. $\lambda = 1$, $\Omega$ has a discrete distribution, and the bidders' signals $X_i$ are distributed exponentially. Expected revenue increases until $Q \approx n/10$, and then falls as $Q$ increases further.

The third and fourth examples strike us as illustrating what we would expect to find in most nonpathological examples: it will pay seller to qualify more than one bidder when there are many bidders, but it will seldom pay to qualify are large subset of the bidders (e.g., a major

CASE 1: Unconditional Independence, Uniform

This case pathologically violates the affiliation assumption driving the Linkage Principle, and is presented to help understand how affiliation affects revenue comparisons, not because of any claims to represent auction markets of interest. Let each $X_i$ be independently and identically distributed as a uniform random variable on $[0,1]$. A straightforward generalization of Harstad, Kagel, and Levin's (1990) proof of revenue equivalence for second-price and English auctions, given unconditional independence, yields revenue equivalence of the $Q$-lottery qualification and $Q$-curtailed oral auctions in this case.

The equilibrium bid function for the $Q$-lottery qualification auction is

$$b_Q(x) = x + \frac{\lambda}{2n} [Q - 1 - x(n - 2)].$$
Let $R_Q^{L_1}$ be the expected revenue for this auction, which calculations show to be

$$R_Q^{L_1} = \frac{(2 - \lambda)n^2 + \lambda(n - 1) - Q(\lambda + 2n(1 - \lambda))}{2n(n + 1)}, \quad (4)$$

which is decreasing with $Q$. So in this case a seller with $n$ bidders would not choose to qualify a subset for a lottery, as opposed to using the auction to find a winning bidder. An interpretation of this result is presented in the next section.

CASE 2: Conditional Independence, Uniform

This case strikes us as less pathological than case 1, in that values are affiliated, and strict revenue comparisons found will hold for approximately uniform distributions.\(^8\) Let $\Omega$ be distributed uniform on $[L, H]$, denote its mean by $\Omega = (L + H)/2$, and given $\Omega = \omega$, let each signal $X_i$ be independently and identically distributed uniform on $[\omega - 1, \omega + 1]$. Equilibrium bids are straightforward:

$$b_Q(x) = x - \lambda \frac{n - 2Q}{n}, \quad (5)$$

$$B_Q^0(x) = x + \frac{\lambda(Q - 1)}{Q + 1}, \text{ and}$$

$$B_Q^0(x | p_1, \ldots, p_k) = \frac{Q(p_1 + 1 - \lambda \frac{Q - 1}{Q + 1}) + x - 1}{p_1} + (1 - \lambda)x. \quad (6)$$

Now the price is set by the bidder holding signal $Z_{Q, i}$, and

$$E[Z_{Q, i} | \Omega = \omega] = \omega + \frac{n - 2Q - 1}{n + 1}.$$

Expected revenue for the $Q$-lottery qualification auction in this case:

$$R_Q^{L_2} \approx \frac{\Omega}{n + 1} + \frac{n - 1}{n + 1} - \lambda + \frac{2Q}{n} (\lambda - \frac{n}{n + 1}), \quad (7)$$
which is increasing in $Q$ if and only if $\lambda > n/(n + 1)$. \textsuperscript{10} In other words, if the efficiency cost associated with qualifying $Q$ bidders for random determination of a winner is sufficiently low, this raises revenue relative to a second-price auction (i.e., $Q = 1$); the revenue-maximizing choice of $Q$ for such nearly-common-value settings is $Q = n - 1$.

This case is typical in that the second-price auction is revenue-inferior to the English auction. However, if $\lambda > n/(n + 1)$ so that larger $Q$'s in the lottery-qualification auction are preferred (in this case), then $Q = 2$ has already overcome the expected revenue advantage of the English auction. Thus, the strength of incentives for a seller to use lottery-qualification can be strong relative to the strength of incentives underlying the revenue comparisons in Milgrom and Weber.

Expected revenue for the $Q$-curtailed oral auction is

$$R_Q^{c_2} = \Omega + \lambda \frac{Q - 1}{Q + 1} - \frac{(1 - \lambda)(2Q + 1 - n)}{n + 1} - \frac{\lambda [(n + 1) - (n - 1)]}{(n + 1)(Q + 1)}.$$

The dependence of this expression on $Q$ is more complicated. A sufficient condition for $R_Q^{c_2} > R_Q^{c_3}$ is

$$\lambda > (Q^2 + Q)/((Q^2 + Q + 1),$$

so a $Q = 2$ curtailed oral auction is revenue-superior to an English auction if $\lambda > 6/7$, and a sufficient condition for $R_Q^{c_2}$ to be strictly increasing in $Q$ is

$$\lambda > (n^2 - n)/(n^2 - n + 1).$$

While $R_Q^{c_2}$ exceeds $R_Q^{c_1}$ for $Q = 1, \ldots, n - 2$, the revenue-maximizing choice is $Q = n - 1$ (for $\lambda$ sufficiently large), for which they are equal. In summary, for this case, when the common-value element is strong enough, seller maximizes revenue by using the lowest bidder as price setter, and then chooses randomly among the remaining bidders to determine a winner, with a significant likelihood of a (mildly) inefficient outcome: the bidder who values the asset most highly has only one chance in $n - 1$ of winning.

**CASE 3: Conditional Independence, Triangular**

Many auction settings might deviate substantially from uniform distributions. Perhaps the most typical deviation would be that signal realizations nearer the mean were more likely than signal realizations further away from the mean. Case 3 incorporates this feature while limiting added calculation difficulties. We calculate bid functions only
for the common-value setting, that is, \( V' = V = \Omega \) (so \( \lambda = 1 \)), and only for the lottery-qualification auction. Let \( V \) be distributed uniform on \([L,H]\), and given \( V = v \), let each signal \( X_i \) be independently and identically distributed with density

\[
  f(x | v) = \begin{cases} 
  x + 1 - v, & v - 1 \leq x \leq v, \\
  v + 1 - x, & v \leq x \leq v + 1. 
  \end{cases}
\]

This triangular distribution complicates matters considerably. While for given \( Q \) and \( n \), \( b_Q(x) = x + r \) with \( r \) rational, a closed functional form \( r(Q,n) \) evades our discovery. Similarly, \( E[Z_{Q-1} | V = v] = v + \rho \) with \( \rho \) rational, but we have had to find \( \rho \) and \( r \) separately for each combination \((Q,n)\). The results are displayed in Table 1, for enough selected combinations to make the pattern clear. Table 1’s entries present approximate expected profit to the lottery winner for the \( Q \)-lottery-qualification

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>.1500</td>
<td>.2000</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
</tr>
<tr>
<td>4</td>
<td>.1158</td>
<td>.1270</td>
<td>.1800</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
</tr>
<tr>
<td>5</td>
<td>.1006</td>
<td>.0940</td>
<td>.1182</td>
<td>.1643</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
</tr>
<tr>
<td>6</td>
<td>.0914</td>
<td>.0764</td>
<td>.0870</td>
<td>.1117</td>
<td>.1512</td>
<td>NM</td>
<td>NM</td>
</tr>
<tr>
<td>7</td>
<td>.0856</td>
<td>.0664</td>
<td>.0685</td>
<td>.0836</td>
<td>.1057</td>
<td>.1400</td>
<td>NM</td>
</tr>
<tr>
<td>8</td>
<td>.0817</td>
<td>.0604</td>
<td>.0571</td>
<td>.0657</td>
<td>.0810</td>
<td>.1000</td>
<td>.1304</td>
</tr>
<tr>
<td>9</td>
<td>.0781</td>
<td>.0567</td>
<td>.0500</td>
<td>.0538</td>
<td>.0645</td>
<td>.0783</td>
<td>.0946</td>
</tr>
<tr>
<td>10</td>
<td>.0753</td>
<td>.0540</td>
<td>.0455</td>
<td>.0458</td>
<td>.0527</td>
<td>.0635</td>
<td>.0755</td>
</tr>
<tr>
<td>11</td>
<td>.0728</td>
<td>.0522</td>
<td>.0426</td>
<td>.0404</td>
<td>.0443</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>.0706</td>
<td>.0406</td>
<td>.0367</td>
<td>.0382</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>.0686</td>
<td>.0393</td>
<td>.0343</td>
<td>.0339</td>
<td>.0376</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>.0668</td>
<td>.0326</td>
<td>.0309</td>
<td>.0329</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>.0651</td>
<td>.0315</td>
<td>.0288</td>
<td>.0293</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>.0635</td>
<td>.0273</td>
<td>.0268</td>
<td>.0288</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>.0620</td>
<td>.0249</td>
<td>.0259</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>.0606</td>
<td>.0236</td>
<td>.0237</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>.0593</td>
<td>.0227</td>
<td>.0220</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: NM = Not Meaningful.
auction with \( Q \) set as indicated at the head of the column, given that \( n \)
indicated at the left of the row is the number of bidders. As this is a
common-value setting, \( E[V] \), expected asset value, is unaffected by
changing \( Q \) or \( n \), so expected revenue figures equal a constant minus the
entries in Table 1.\(^{11}\) For each \( n \), the revenue-maximal \( Q \) is indicated by
a double-bordered entry. As second-price auctions are the special case \( Q = 1 \), the column labeled (2-P) provides a comparison with single-stage
auctions.

A fairly regular pattern arises: expected revenue first rises, then falls,
as \( Q \) is increased, with the revenue-maximal \( Q \) near \( n/3 \). For \( n \geq 5 \), it pays
to qualify multiple bidders for a lottery rather than run a second-price
auction. While only the common-value case is presented in Table 1, strict
inequalities will also apply when \( \lambda \) is close to 1. For example, with eight
bidders, \( \lambda > .98 \) is sufficient for \( Q = 3 \) to be best, and \( Q = 2 \) is best for
\( .88 < \lambda < .98 \); for lower values of \( \lambda \) the efficiency loss associated with
lottery qualification is too costly.

CASE 4: Discrete Values, Exponential Signals

Since the bid functions in Cases 2 and 3, while continuous, are not
smooth at \( x = L + 1 \) and \( x = H - 1 \), we now consider the following
example: given \( V = \nu \), each \( X_i \) is exponentially distributed, with condi-
tional density \( (1/\nu)e^{-x/\nu} \) that \( X_i = x \). Also for tractable contrast, the
underlying distribution of \( V \) is nonuniform: \( V = \frac{1}{2} \) with probability \( \frac{2}{5} \),
and \( V = 1 \) with probability \( \frac{1}{5} \). For these parameters,

\[
b_0(x) = \lambda \frac{4(1 - e^{-(2\nu)Q+1}e^{-2\nu Q+1}) + (1 - e^{-\nu Q+1}e^{-\nu Q+1})}{8(1 - e^{-(2\nu)Q+1}e^{-2\nu Q+1}) + (1 - e^{-\nu Q+1}e^{-\nu Q+1}) + (1 - \lambda)x} \quad (8)
\]

is the symmetric equilibrium bid function. The expected revenue formula
derived from (8) is impenetrable, so following the pattern of Case 3,
Table 2 presents enough calculations to make the expected revenue
pattern clear in the common-value case (analogous to Table 1, expected
revenue is \( \frac{2}{5} \) minus \( \frac{1}{10} \) of the amounts shown.\(^{12}\) This case roughly
coincides with the previous one, though it takes more competition before
qualifying multiple bidders begins to pay off: \( Q = 2 \) becomes revenue-
superior to \( Q = 1 \) for \( n \geq 10 \). As \( n \) increases, the revenue-maximal \( Q \)
increases slowly, approximated by \( (n - 1)/5 \) over most of the range we
have calculated. Table 3 summarizes: for each \( Q \), it specifies \( n \), the
minimum \( n \) for the seller to prefer to qualify \( Q \) bidders.
Table 2. Mean Winner’s Profit, Case 4

<table>
<thead>
<tr>
<th>n</th>
<th>Q</th>
<th>π</th>
<th>Q</th>
<th>π</th>
<th>Q</th>
<th>π</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>.5549</td>
<td>2</td>
<td>.6045</td>
<td>3</td>
<td>NM</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>.4809</td>
<td>2</td>
<td>.5096</td>
<td>3</td>
<td>.5320</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>.4274</td>
<td>2</td>
<td>.4409</td>
<td>3</td>
<td>.4688</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>.3862</td>
<td>2</td>
<td>.3885</td>
<td>3</td>
<td>.4058</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>.3689</td>
<td>2</td>
<td>.3666</td>
<td>3</td>
<td>.3796</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>.3142</td>
<td>2</td>
<td>.2986</td>
<td>3</td>
<td>.2991</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>.3032</td>
<td>2</td>
<td>.2852</td>
<td>3</td>
<td>.2833</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>.2666</td>
<td>3</td>
<td>.2327</td>
<td>4</td>
<td>.2328</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>.2588</td>
<td>3</td>
<td>.2223</td>
<td>4</td>
<td>.2211</td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>.2383</td>
<td>4</td>
<td>.1911</td>
<td>5</td>
<td>.1919</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>.2322</td>
<td>4</td>
<td>.1825</td>
<td>5</td>
<td>.1824</td>
</tr>
<tr>
<td>28</td>
<td>1</td>
<td>.2107</td>
<td>5</td>
<td>.1506</td>
<td>6</td>
<td>.1510</td>
</tr>
<tr>
<td>29</td>
<td>1</td>
<td>.2060</td>
<td>5</td>
<td>.1438</td>
<td>6</td>
<td>.1437</td>
</tr>
<tr>
<td>33</td>
<td>1</td>
<td>.1891</td>
<td>6</td>
<td>.1191</td>
<td>7</td>
<td>.1193</td>
</tr>
<tr>
<td>34</td>
<td>1</td>
<td>.1853</td>
<td>6</td>
<td>.1138</td>
<td>7</td>
<td>.1136</td>
</tr>
<tr>
<td>38</td>
<td>1</td>
<td>.1716</td>
<td>7</td>
<td>.0943</td>
<td>8</td>
<td>.0946</td>
</tr>
<tr>
<td>39</td>
<td>1</td>
<td>.1685</td>
<td>7</td>
<td>.0904</td>
<td>8</td>
<td>.0903</td>
</tr>
</tbody>
</table>

Notes: Entries are 10 times winner’s expected profit. NM = Not meaningful.

In the absence of affiliation between bidders’ information, Case 1 maximizes expected revenue at \( Q = 1 \). Case 2 shows that a widely used parametric example exhibits increased revenue as privacy is further undermined, all the way to \( Q = n - 1 \). Cases 3 and 4, however, indicate that frequently qualifying only a smaller fraction of bidders is revenue-maximal, and that the best number to qualify can depend on specific distributional features. Experimentation with varying \( Q \) may be called for in any field implementation.

Table 3. Case 4: Minimum Number of Bidders to Justify Qualifying \( Q \) Bidders

<table>
<thead>
<tr>
<th>Q</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15</td>
<td>20</td>
<td>24</td>
<td>29</td>
<td>34</td>
<td>39</td>
<td>44</td>
<td>49</td>
<td>54</td>
<td>59</td>
<td>63</td>
<td>68</td>
<td></td>
</tr>
</tbody>
</table>
IV. INTUITION UNDERLYING REVENUE COMPARISONS

Milgrom and Weber (1982) explain that their revenue comparisons share a common theme, the “Linkage Principle”:

To the extent that the price in an auction depends directly on variables other than the winning bidder’s report, and to the extent that these other variables are (at equilibrium) affiliated with the winner’s value estimate, the price is statistically linked to that estimate. The result of this linkage is that the expected price paid by the bidder, as a function of his estimate, increases more steeply than it otherwise might. . . a steeper payment function yields higher prices (and lower bidder profits).

In the first-price auction, for example, revealing the seller’s information links the price to that information, even when the winning bidder’s report \( x \) is held fixed. In the second-price auction, the price is linked to the estimate of the second-highest bidder, and revealing information links the price to that information as well. In the English auction, the price is linked to the estimates of all the non-winning bidders, and to the seller’s estimate as well, should he choose to reveal it. The first-price auction, with no linkages to the other bidders’ estimates, yields the lowest expected price. The English auction, with linkages to all of their estimates, yields the highest expected price (Milgrom and Weber, 1982, pp. 1110–1111).

The Linkage Principle is more formally introduced as a complex mathematical proposition in Milgrom (1987, Prop. 6, p. 20), under assumptions that render it silent about such nonstandard mechanisms as lottery-qualification auctions. However, the same paper’s introduction reinterprets the principle more broadly:

Intuitively, a bidder’s expected profits from an auction are greatest when he privately knows a high value, and the item being sold is quite valuable. The intuition of the Linkage Principle is that the auctions yielding the highest average prices are those most effective at undermining the privacy of the winning bidder’s information, thereby transferring some profits from the bidders to the seller. According to the principle, privacy is undermined by linking price to information other than (but correlated with) the winning bidder’s private information (Milgrom, 1987, p. 4).

 Essentially, the highest-valuing bidder’s information is “undermined,” that is, becomes less valuable to him, because the information of his most serious competitor becomes a closer substitute for his own information. Thus, the difference between his information and the closest substitute for it yields less profit. Milgrom (1985) explicitly claims that the Linkage Principle explains why second-price auctions generate more revenue than first-price auctions; Riley (1989) disagrees that the principle supplies the right intuition. Riley’s concerns are reinforced by the
fact that the Linkage Principle cannot directly be extended to account for revenue comparisons in Cases 3 and 4 above.

Lottery qualification auctions go further in the direction of undermining information privacy, in the particular sense that the private information of each other qualifier becomes a perfect substitute (i.e., yields identical expected profit) for the highest-valuing bidder’s private information. Indeed, the highest-valuing bidder has no strategy available to him that can yield him any greater expected profit than the other qualifiers. It would appear that the greater the number of qualifiers, the more diluted the highest-valuing bidder’s private information has become.

But Cases 3 and 4 above show that undermining privacy in this sense and augmenting expected revenue do not go hand in hand. This section offers the intuition we believe lies behind lottery qualification auction revenue comparisons; readers may find it also adds to understanding the impact of public information in standard auctions and the revenue comparisons between second-price and English auctions.

For concreteness, we focus on the common-value case ($\lambda = 1$); the same issues arise more generally. The equilibrium bid function $b_Q$ is a conditional expectation; it expresses a price at which the bidder is indifferent between qualifying and not qualifying for the lottery. There is in general no one such price of indifference: if rivals’ bidding is changed so that paying a previous price of indifference $p$ now yields more positive inferences about rivals’ signals, the new price of indifference is higher. However, in symmetric, monotonic equilibrium, to assume that the $Q$th highest rival signal, exceeds your signal is to assume that you have failed to qualify. So the highest price of indifference consistent with symmetric equilibrium is $E[V \mid X = x, \tilde{Y}_Q = x] = \nu_Q(x, x) = b_Q(x)$.

This expectation will only be borne out in the (zero probability) event of a tie for last qualifier. Barring that event, every bidder is bidding an incorrect estimate of $V$: on being told that he qualified (or even that he was the last qualifier), a bidder’s correct Bayesian estimate of $V$ will be less than his bid (but more than the price), because he will no longer assume a tie. On being told that he did not qualify, his estimate of $V$ will be more than his bid (but less than he would have had to bid in order to qualify), again because he no longer assumes a tie.

Consider in particular the price setter, who observed the $(Q + 1)$st highest signal, $\tilde{Z}_{Q+1}$; having rationally bid as if it mattered to him what he bid, his bid has estimated $V$ as if his signal were both the $Q$th and $(Q + 1)$st highest signals, underestimating since $\tilde{Z}_{Q+1} < Z_Q$. His bid’s
underestimate is the source of expected profit to the lottery qualifiers; recall that lower expected profit corresponds to higher expected revenue.\(^{14}\)

While conceptually equivalent to considering how the price setter estimates asset value, \(V\), it is more illuminating (for understanding the cases above) to focus on the price setter’s estimates of his rivals’ rank-ordered signals \(Z_i\). Admittedly, this would be a circuitous method of estimating \(V\).\(^{15}\) The following shorthand is used: \(z_i = E[Z_i]\), which will coincide (on average) with the expectation that the price setter would make ex post—on learning that he is the price setter. In contrast, \(z_i\) will denote the expectation of \(Z_i\) which the price setter incorporates in his bid.

Differences in expectations in Cases 1 and 2 are simplified below by a parameter \(\kappa = \frac{1}{2n(n + 1)}\). For concreteness, most of the discussion will presume \(n = 5\), so \(\kappa = 1/60\).

**CASE 1: Lottery-Qualification Auction Intuition**

Figure 1 analyzes the simplest situation, Case 1, showing why lottery-qualification auctions are revenue losers there. At the top are shown the average levels of the second- and third-highest signals. The \(Q = 1\) line illustrates the average occurrence in the second-price auction. In assuming his signal was both highest and second-highest, the price setter, observing on average \(Z_2\), bids as if \(Z_1 = Z_2 = Z_2\). On average, however, he

---

**Figure 1.** Case 1
does not distort the estimate of lower signals. Thus, the winner’s expected profit results solely from the effect of the 10k reduction in \( z_1 \) rationally assumed for bidding purposes. When \( Q = 2 \), the price setter observes \( z_1 \) on average, and effects analogous to those just mentioned arise: lower signals are undistorted, and the immediately higher signal is assumed lower by an average of 10k. However, an additional effect diminishes expected revenue: \( z_1 \), now assumed the only strictly higher signal, is assumed on average to be halfway between \( z_3 \) and 1; on learning that he set the price, this bidder would revise that estimate up 5k. This distortion of signals ranked more than one rank higher than the price setter’s is the sole impact of changing \( Q \) upon revenue, and becomes exacerbated the more such signals there are.

**CASE 2: Lottery-Qualification Auction Intuition**

The pathological nature of this aspect of Case 1 stands in sharp contrast to the corresponding explication of Case 2, illustrated in Figure

\[ Q = 1: \]

\[ Q = 2: \]

\[ Q = 3: \]

*Figure 2. Case 2*
2. We rely on the maximum and minimum of a sample from an unknown-location uniform distribution being jointly sufficient statistics for the entire sample: any distortions of $Z_i$ for $i$ other than 1 or $n$ are irrelevant to calculating the equilibrium bid. Again, in the second-price auction ($Q = 1$), the price setter bids as if $z_i = z_1$, corresponding to a profit-producing distortion of $20\kappa$ (the horizontal scale has been halved). However, the presumption of two identical signals will suggest that the remaining signals are more widely spaced on average, so it also matters that the proper Bayesian estimate of $Z_2$ incorporated in bidding is $12\kappa$ less than the proper estimate upon learning that you are the price setter.

When $Q = 2$, the price setter observes on average $z_3$, and bids as if $z_2$ were also equal to his signal; this effect is not directly comparable to the effect when $Q = 1$, as the only rival signals whose expectation influences the bid are the extreme signals. As just mentioned, a bidder who assumes one rival signal equals his will then expect the remaining rival signals to be more widely spaced on average. However, this leads the bidder who assumes his signal to be both second- and third-highest to place $z_3$, his resulting conditional expectation of the lowest signal, closer to $z_2$ ($8\kappa$) than did the price setter in $Q = 1$ above ($12\kappa$), as now there is only one signal between the two assumed tied and the lowest. A similar effect appears in the estimate of $Z_1$, now a signal assumed more widely spaced than before, only the wider spacing works to place the estimate of this signal higher, now only $16\kappa$ below the price setter’s ex post estimate, rather than $20\kappa$. Both differences found when $Q = 2$ lead to sufficiently more aggressive bidding as to overcompensate for using a lower rank-ordered bid to set the price, so the winner’s expected profit is less and expected revenue is greater.

The same pattern continues when we move to $Q = 3$: the lowest signal is now adjacent to the two signals assumed tied, so the assumed wider spacing matters less (only $4\kappa$), while wider spacing applies to both signals assumed higher so that $Z_1$ is only distorted by $12\kappa$. If we continued to $Q = 4$ (not shown), the pattern would continue, yielding no distortion of the lowest signal (which is in fact the signal observed by the price setter) and an $8\kappa$ distortion of the highest signal.

**CASES 1 and 2: Curtailed Oral Auction Intuition**

Next, let us briefly use Figures 1 and 2 to illustrate what happens in the $Q$-curtailed oral auction. In Case 1, on average, signals below the price setter are not distorted by the assumption of tied signals, so on
average the revelation of those signals by an oral auction process would not alter revenue. In Case 2, the distortions of the lowest signal indicated in Figure 2 would be eliminated in a curtailed oral auction, as the price at which the first bidder quit would reveal $z_0$. Accordingly, the price setter, in selecting his bid, rationally assumes that the signals below his are more closely spaced (since he has learned the lowest signal and presumes his to be both $Q$th and $(Q + 1)$st highest). This leads him to assume that signals above his are even more widely spaced than shown in Figure 2, moving $z_1$ even closer to $Z_1$, hence increasing expected revenue, both relative to the $Q$-lottery qualification auction (for $Q < n - 2$) and to $Q$-curtailed oral auctions with fewer qualifiers.

**CASE 3: Lottery-Qualification Auction Intuition**

With the triangular distribution, neither the unweighted mean (as in Case 1) or the extreme order statistics (as in Case 2) form a sufficient statistic for the collection of order statistics in estimating $V$. However, there are a set of weights $w_i, i = 1, \ldots, n$ which are the coefficients of the minimum-variance linear unbiased estimator of $V$ from the order statistics $(Z_1, \ldots, Z_n)$ (David, 1970, §6.2). These coefficients are most compactly denoted as the first row of the matrix $\Theta$:

$$\Theta = (A' C^{-1} A)^{-1} A'$$

where

$$A = \begin{bmatrix} 1 & E[Z_1] \\ \vdots & \vdots \\ 1 & E[Z_n] \end{bmatrix}$$

and $C$ is the covariance matrix of the order statistics. Naturally, these weights are symmetric, placing equal reliance on order statistics $i$ and $n + 1 - i$. They show a similar pattern across various numbers of bidders, with the highest weights placed on the extreme order statistics, because the compact, ordered support makes these extremes relatively indicative. The remaining order statistics are weighted in respect to their proximity to the median. Figure 3 illustrates the distortions and their relative importance, presuming $n = 5$, corresponding to the previous figures. The weights are indicated by the heights of the rectangles. (The weight attached to an undistorted signal is shown as a vertical line.) Figure 4 is the comparable illustration for $n = 8$. 
Figure 3. Case 3, \( n = 5 \)
$Q = 1$: $v - 1 \leq Z_2 = Z_1 Z_1 \leq v + 1$

$Q = 2$: $v - 1 \leq Z_3 = Z_2 Z_2 \leq v + 1$

$Q = 3$: $v - 1 \leq Z_4 = Z_3 Z_3 \leq v + 1$

$Q = 4$: $v - 1 \leq Z_5 = Z_4 Z_4 \leq v + 1$

*Figure 4.* Case 3, $n = 8$
Why does expected revenue stop rising with $Q$ at $Q = n/3$, rather than continuing to rise at least until $n/2$? The price setter, observing $Z_{Q-1}$ by assuming that his signal is both $Z_Q$ and $Z_{Q-1}$, is only mildly distorting signals below his by this assumption, but is attaching an ordinally incorrect rank to signals higher than his. For example, assuming $Z_{Q-1}$ is the immediately adjacent signal. Thus signals $(Z_1, \ldots, Z_{Q-1})$ are much more heavily distorted than $(Z_{Q-2}, \ldots, Z_n)$, and including more signals in the first category starts to interfere with the ceteris paribus assumption of the previous paragraph.

In Case 4, similarly, moving to a lower-variance order statistic to set the price is profitable, given enough bidders. But this effect is even more limited by the serious impact of attaching an ordinally incorrect rank to higher signals. Presumably the ordinally incorrect rank has a greater impact than in Case 3 because higher signals have unbounded supports.

V. CONCLUSIONS

When bidders are fully aware that competition among them will be curtailed, it may pay the seller to do so. In Case 2 above, where both the underlying common-value variable and bidders' private information signals are uniformly distributed, increasing $Q$, the number of the $n$ bidders whose bids simply qualify them for a lottery awarding the asset, increases expected revenue if the asset has a sufficiently strong common-value aspect. The seller's best choice in that case is simply to set the price equal to the lowest bid submitted, and choose a winning bidder equi-probably among those whose bids were not the lowest. In Cases 3 and 4, which we consider more realistic, qualifying too many bidders is unwise, but it pays to qualify more than just the highest bidder: $Q$ is best set at the integer nearest $n/3$ (Case 3) or $(n-1)/5$ (Case 4). Whenever $Q < n - 1$, seller is better off curtailing competition orally, rather than with sealed bids, and should publicly reveal any information he possesses which is affiliated with asset value.

We suggest that a rather natural intuition is at work behind these revenue comparisons, one that may also illuminate other revenue comparisons across auction procedures selling to $n$ bidders. Bidders rationally focus on payoff-relevant events; this leads the bidder who will set the price to distort the estimate of asset value he would make ex post (on finding out he was the price setter). The lesser is this rational distortion, the greater is expected revenue. This logic handles our more complicated cases, to which the intuition behind the Linkage Principle fails to extend.
Our paper points to new reasons for looking at multistage bargaining and transacting procedures. Multistage procedures may provide dimensions in which to undermine privacy of competitors' information that would be omitted in single-stage analyses, and may offer useful insights into when undermining information privacy is an appropriate objective. The qualitative results obtained here may extend to second-stage decision rules in which aspects of the final contract are determined within stages as follows: the first stage sets price and determines both the subset of bidders willing to supply at that price, and seller's share of gains from trade. The second stage determines non-price aspects of the contract in such a way that the ultimately successful contracting partner appears to be random, as viewed ex ante.

Antecedent work, by focusing on standard auctions, has also cast potential allocative inefficiencies in too limited a light. Given an underlying symmetric situation, the literature has suggested that inefficient allocations only result when seller strategically commits to a nontrivial reserve price. A sale, if it occurs, has always been viewed as going to the bidder valuing the asset most highly. When a seller begins to look in the directions introduced here, sales to symmetric bidders may also be inefficient, though sizable inefficiencies are likely to be costly to the seller.

Why do we not see lottery-qualification or curtailed oral auctions in markets regularly transacting by auction? It is worth returning to our introductory caution about assuming rational behavior, and noting that experienced bidders may exhibit a disequilibrium phase of bidding substantially less aggressively than called for in equilibrium. Also, the complications introduced by assuming asymmetry in bidder information or evaluation are more serious for lottery-qualification or curtailed oral auctions than for second-price or English auctions (cf. Harstad, 1991). Moreover, a seller's credibility is typically taken for granted in the auction literature; this assumption may be more seriously astray here than usually: if, say, \( n = 10 \) and \( Q = 5 \), a bidder needs to be extremely confident that he faces nine independent competitors and a seller will honestly give the five highest bidders an even shot at buying at the sixth highest bid before that bidder will be willing to make the extremely high bids necessary to reach equilibrium revenue. If the sixth highest bid is far below the fourth highest, bidders must believe the seller will resist the temptation to introduce two fictitious qualifying bids, and then rig the lottery to award to one of the three legitimate bids that qualified.
Our conclusions have come from analyzing a single, isolated auction; in practice, auction markets typically transact many related assets in a single session, simultaneously or sequentially. Extending this analysis to such settings requires caution. \(^{18}\) We close by mentioning two of many possible concerns. One is that winning bidders may have to transact with third parties. Rothkopf, Teisberg, and Kahn (1990) suggest that winners in a second-price auction may be at a disadvantage in later negotiations with third parties if the amount of rent earned in winning the auction is revealed by the difference between their bid and the price paid. This concern could be sharper in lottery-qualification auctions although the mechanism obviously calls for bids in excess of willingness to pay.

Secondly, we have followed common practice in employing the artifice of analyzing revenue-relevant changes in auction procedures while holding the number of bidders fixed. Realistically, if increasing \(Q\) led to greater equilibrium revenue given \(n\) bidders, that is, if increasing \(Q\) was a device to extract more surplus from \(n\) bidders, then increasing \(Q\) will actually lead to fewer bidders participating. This tradeoff is considered in Harstad (1993) for standard auctions. His findings likely extend to lottery-qualification auctions: when potentialidders are numerous, information-gathering costs slight, and precise asset value estimates impossible, the impact of fewer participants will be less than the surplus extraction impact, and the revenue comparison for a fixed number of bidders will be extended; in the opposite circumstances, the amount of participation will be the more important variable, and the revenue comparisons above would be reversed.

**APPENDIX**

1. Proof of Proposition 1

Since \(b_{Y_Q}\) is increasing, the price bidder 1 faces if he qualifies is \(b_{Y_Q}(Y_Q)\), and his conditional expected payoff when he bids \(b\) is

\[
E \left( \frac{1}{Q} \left[ V_{1} - b_{Y_Q}(Y_Q) \right] 1_{\{b,Y_Q \leq b\}} | X_1 = x \right)
\]

\[
= \frac{1}{Q} E \left[ E \left( [V_{1} - v(Y_Q,Y_Q)] 1_{\{b,Y_Q \leq b\}} | X_1 = x \right) \right]
\]

\[
= \frac{1}{Q} E \left( [v_{Y_Q}(X_1,Y_Q) - v(Y_Q,Y_Q)] 1_{\{b,Y_Q \leq b\}} | X_1 = x \right)
\]
\[ \frac{b_Q^{(b)}}{Q} = \frac{1}{Q} \int_{-\infty}^{b_Q^{(b)}} [v_Q(x, \alpha) - v_Q(\alpha, \alpha)]f_Y(\alpha \mid x)\,d\alpha. \]

where \( f_Y(\cdot \mid x) \) is the conditional density of \( Y_Q \) given \( X_1 = x \). Since \( v_Q \) is increasing in its first argument, the sign of the integrand matches the sign of \((x - \alpha)\). Hence the integral is maximized by choosing \( b \) so that \( b_Q^{(b)} = x \), and \( b_Q^{(b)} \) is a best reply. The profile \( (b^Q_1, \ldots, b^Q_n) \) has been shown to be a symmetric Nash equilibrium; a proof of uniqueness can be obtained by altering the argument in Levin and Harstad (1986) in precisely the same way Milgrom and Weber's argument has just been altered; details are omitted.

2. Proof of Proposition 2

It is straightforward to verify that \( B^Q \) is increasing in its first argument, for each \( Q \leq n - 1 \) and \( k \leq n - Q - 1 \). Suppose bidders \( 2, \ldots, n \) adopt \( B^Q \). If bidder 1 qualifies, he will with probability \( 1/Q \) win and thus pay

\[ \mathbb{E}[V_1 \mid X_1 = y^Q, Y_Q = y^Q, \ldots, Y_{n-1} = y_{n-1}], \]

where \( y^Q, \ldots, y_{n-1} \) are realizations of \( Y_Q, \ldots, Y_{n-1} \). His conditional estimate of \( V_1 \) given \( X_1, Y_Q, \ldots, Y_{n-1} \) is

\[ \mathbb{E}[V_1 \mid X_1 = x, Y_Q = y^Q, \ldots, Y_{n-1} = y_{n-1}], \]

so his conditional expected payoff is nonnegative if and only if \( x \geq y_Q \). Using \( B^Q \), bidder 1 will qualify if and only if \( X_1 > Y_Q \) (recall \( \{X_1 = Y_Q\} \) is null). Thus, \( (B^Q_1, \ldots, B^Q_n) \) is a Nash equilibrium.

ACKNOWLEDGMENT

We thank Jacques Crémer, Doug Davis, Shannon Mitchell, Dan Newlon, Mike Rothkopf, and especially Dan Levin for helpful comments. Support from National Science Foundation grant SES 91-08551 is gratefully acknowledged. Remaining errors are solely our responsibility.

NOTES

1. To complete the description of auction rules, if two or more bidders tie for the \( Q \)th highest bid, the \( Q - 1 \) highest bidders qualify with probability 1, and an anonymous random method selects one more qualifier from those who tied; the bid level at which
they tied becomes the price. Such ties will have zero probability below. Some analysis below surely has a counterpart if the \( Q \)th highest bid determined the price. It this would be mathematically much more cumbersome, and has not been explored by us. Given \( n \) bidders, and \( Q \) qualifiers, presumably a close analogue to Theorem 15 in Milgrom and Weber (1982) would show that the seller prefers to use the \( (Q + 1) \)st highest bid to set the price.

2. A new oral ascending auction form corresponding to lottery-qualification auctions, the "\( Q \)-curtailed oral" auction, is also considered below. In this example, it also attains revenue superiority to standard auctions. We should mention that a highly artificial and foreign allocation mechanism presented by McAfee, McMillan, and Reny (1989) would extract nearly all the surplus in a pure common-value environment, attaining expected revenue \( P \) in this example, if one assumes that bidders acquire their private estimates costlessly, and are willing to participate in a foreign mechanism even when the ex ante profitability of doing so is zero.

3. We should emphasize that the laboratory studies cited are not directly relevant, in that all laboratory auctions of which we are aware use standard auction forms. However, subjects' tasks in a lottery qualification auction would seem to be at least as complex as in the cited studies.

4. Affiliation, which also appears in the statistical literature as the monotone likelihood ratio property, and by other names, is discussed in Milgrom and Weber (1982, pp. 1098–1100). Roughly speaking, a pair of random variables are affiliated if a higher value for one makes higher values for the other more likely. Independence is a special case. Exchangeability means ex ante symmetry.

5. The common-value setting (Wilson, 1977) is the special case where \( V_i = S_i \). This setting generalizes to include the symmetric possibility that the asset's value includes both common and bidder-specific attributes.

6. Harstad and Rothkopf (1991) argue that bidders in English auctions typically have the will and the way to prevent rivals from observing such complete information. Equilibrium expected revenue in a formulation provided in Harstad and Rothkopf (1991) is between second-price revenue and the revenue of the Milgrom-Weber formulaic English auctions. In the Harstad-Rothkopf formulations, it may not be clear when to curtail an oral auction so as to have precisely \( Q \) remaining competitors, since a bidder’s decision to cease competing is realistically modeled as temporary and reversible, and may not be publicly observable behavior.

7. A similar logic can also extend to lottery-qualification auctions the results of Riley (1988), showing that the present value of expected revenue can be enhanced by collecting some of the revenue in the form of royalties, rather than all in a one-time payment.

8. Section II's results indicate that equilibrium bid functions are conditional expectations, so both bid functions and order statistics can be expressed as continuous functions of some parametric departure from the uniform distribution. While Case 1 could also be generalized to approximate uniformity and slight parametric deviations from unconditional independence, the latter deviations seem insufficient to render that case plausible.

9. These formulas designate equilibrium bid functions for realizations \( x \in [L + 1, H - 1] \), for which \( X_i \) is an unbiased estimate of \( V_i \). Extensions to lower and higher signal realizations involve the following complications. First consider \( x \in [L - 1, L + 1] \); arguments in Milgrom and Weber (1982) and Kagel, Levin, and
Harstad (1989) can be adapted to this setting to show that, for such realizations, $Z_i$ is a
~garbling~ $\rightarrow Z_i$, and a zero profit condition characterizes $b_Q$. Then consider
$x \in [H - 1, H + 1]$; here $v_Q(x, x)$ does not have a closed form solution (as a function of
$Q$, see Kagel, Levin, and Harstad, 1995 for the $Q = 1$ case), but the formula given by (6)
has been shown (in an appendix to Kagel, Levin, and Harstad, 1989) to offer an acceptable
approximation. Bear in mind that these approximation issues at endpoints do not carry
over into $[L - 1, H - 1]$; here (6) is precisely the formula for $b_Q$ as defined above.

10. This approximation results from extending the exact calculation for
$[L + 1, H - 1]$ to the entire domain (for which the bid function is no longer exact, see
note 9). This same approximation method will be used for the curtailed oral auction, and
in Case 3. Its accuracy for strict revenue rankings depends on a bidder observing a signal
being sufficiently better informed than a bystander who has no private information, which
in these cases translates into $(H - L)$ being large relative to 1. A crude bound on the
accuracy of this approximation can be obtained by presuming that a revenue-superior
mechanism is extended to $[H - 1, H + 1]$ via a horizontal bid function, while a revenue-
inferior mechanism is extended via a zero profit condition to this region. This yields the
bound that a revenue comparison presented is accurate to within $10^{-8}$ for $(H - L - 2) > 0.667 \cdot 10^6$. Thus, a crude bound for revenue to increase with $Q$, in this example with
$\lambda = 1$, is $(H - L - 2) > n(n + 1)/3$. An example auction in which $(H - L)$ is small is the
combined sale of a Treasury bond maturing Friday and a fistful of pennies. It is hard to
imagine an auction of economic interest which one would want to stylize by making
$(H - L)$ small.

11. Calculations were performed via Wolfram Research's Mathematica symbolic
logic interpreter. Integrations and simplifications maintained arbitrary precision, attaining
exact rational results for $x \in [L + 1, H - 1]$. so that extension to the full domain (as
discussed in notes 9 and 10) is the only source of approximation. Hence, the bound on
approximation errors mentioned in the previous note applies. Formulas used by Mathematica
for Case 3 are available from the authors on request.

12. Again, the revenue-maximal $Q$ for the $n$ heading each row is shown as a
rebordered entry. As (8) is exact, numerical approximation only enters the integration
routines used; expected revenues have been calculated to within $10^{-16}$.

13. Recall this is not a challenge to the mathematical correctness of the Linkage
Principle, as Milgrom (1987) rules out nonstandard auctions before defining the principle.

14. Focusing on the price setter's rational underestimate can also shed light on
revenue comparisons across standard auction forms. An English auction raises more
revenue than a second-price auction because in the latter the price setter, in assuming
payoff relevance, distorts (rationally underestimates) lower signals which are revealed
in the course of an English auction (and thus unaffected by this distortion). Similarly, a
seller by revealing public information diminishes both remaining asset value uncertainty
and, accordingly, the amount of distortion the price setter incorporates in his bid. Linking
the price to exogenous affiliated indicators of asset value (including those revealed ex
post, as in Riley, 1988) also serves to diminish the impact of the price setter's rational
distortions.

15. Though this involves an extra step in the calculation, it is entirely consistent with
the logic of the equilibrium. In the Introduction's example (Case 2 with $n = 3$), the
second-price bidder assumes his signal tied for highest. He can then calculate that, on
average, \( F \) will be 1/3 less than his signal. Equivalently, he can estimate that, on average, the remaining signal will be 2/3 less than his signal, and \( F \) will on average be located at the midpoint of his and the lowest signal.

16. As in the calculations, low probability events where bidders observe biased signals due to endpoint effects are ignored.

17. Technically, this method is being used to determine the weights in all three cases, as he first row of \( \Theta \) is \( (1/n, \ldots, 1/n) \) for Case 1, and \( (1/2, 0, \ldots, 0, 1/2) \) for Case 2.


REFERENCES

