A Class of Dominance Solvable Common-Value Auctions

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Dominant strategies seldom exist in non-cooperative games. Moulin's concept of a dominance solvable game generalizes, dominant strategy without dramatic loss in appeal. We consider a class of common-value auctions characterized by the property that the maximum of a collection of informative signals is a sufficient statistic for the entire collection. We demonstrate that this class of second-price auctions is dominance solvable.

1. INTRODUCTION

In the independent private values model, Vickrey (1961) analysed auctions where an indivisible item had a value to each of \( n \) bidders that was an independent draw from a known probability distribution. Bidders simultaneously submitted sealed bids, each knowing only his valuation of the item when bidding. Vickrey's most striking result was that bidding this value is a dominant strategy in a second-price auction.\(^1\)

Dominant strategies seldom exist in non-cooperative games. The appeal of a dominant strategy is straightforward: introspection is sufficient for its adoption, and the need for coordination and/or feedback in order to reach equilibrium is eliminated. Absent a dominant strategy, noncooperative models lack dynamic structure and stability analysis; this is a most difficult gulf to bridge.

Moulin's concept of a dominance solvable game generalizes dominant strategy without dramatic loss in appeal.\(^2\) Again, it is a strong requirement seldom met in games, expecting repeated application of a process of deleting dominated strategies to yield a unique outcome. In a dominance solvable game, a weak presumption about the rationality of other players forms a basis for introspective arrival at the predicted behaviour.

Common value auctions, first studied by Wilson (1977), are more complex and realistic than independent private value auctions. The value of the item is, to a first approximation, common across bidders but unknown when bids are submitted. Vickrey's argument for a dominant strategy does not extend to common-value auctions, as it depends critically upon a bidder's certainty about the item's value to him (Milgrom and Weber (1982), p. 1091). Bidders only observe informative signals correlated with the item's value, and generally face both strategic and evaluative problems.

We consider a class of common-value auctions characterized by the property that the maximum of a collection of informative signals is a sufficient statistic for the entire collection. This statistical property is a plausible description of certain auction markets, and can readily be satisfied in laboratory economies. We demonstrate that this class of second-price auctions is dominance solvable in two steps.
2. NOTATION AND ASSUMPTIONS

An indivisible item is auctioned to \( n \) bidders under second-price sealed-bid rules. If bidder \( i \) wins and pays price \( p_i \), the value to him of this outcome is \( u(V - p_i) \), where \( V \) is drawn from a distribution \( G(V) \) with density \( g(V) \), and \( u \) is increasing and concave with \( u(0) = 0 \). Information known about \( V \) when bids are formed is represented by \( X = (X_1, \ldots, X_n) \), where \( X_i \) is the real-valued informative signal observed (privately) by bidder \( i \). \( V \) and \( X \) are jointly drawn from distribution \( F(X, V) \) with density \( f(X, V) \) and conditional density \( h(X|V) \). We assume \( E[V] < \infty \), and maintain the important assumption that the variables \( (V, X_1, \ldots, X_n) \) are affiliated.\(^4\) \( E[u(V|X_i = x)] \) is assumed an increasing function of \( x \).\(^5\)

Finally, \( h(X|V) \) is assumed to be symmetric. All signals are distributed symmetrically, so rational bidders treat signals symmetrically in evaluating \( V \). Symmetry can be exploited to focus on bidder 1, so notation \( Y_1, \ldots, Y_{n-1} \) represents signals \( X_2, \ldots, X_n \) arrayed in descending order.

A strategy for player \( i \) is a function \( b_i(x_i) \) providing a bid when \( X_i = x_i \) is observed. Regard strategies \( b_i \) of bidders \( j = 2, \ldots, n \) as fixed. The highest bid among them is \( p := \max_{j \neq i} b_j(x_j) \), a random variable. Bidder 1 wins at price \( p \) if \( b_1 > p \). Thus, his rational behavior is

\[
\max_{b_i} E[u(V - p)|b_1 > p, X_1 = x],
\]

where \( 1_{b_1 > p} \) takes the value 1 in event \( (\cdot) \), 0 otherwise. Standard definitions: \( b_1(x_1) \) is a best response to \( (b_2, \ldots, b_n) \) if it solves (1) for all \( x \); \( (b_1, \ldots, b_n) \) is a Nash equilibrium if \( b_i \) is a best response to \( (b_j)_{j \neq i} \) for all \( i = 1, \ldots, n \); \( b_1 \) is a dominant strategy if it is a best response for every opposition \( (b_2, \ldots, b_n) \).

3. DOMINANCE SOLVABILITY

Let the function \( v(x, y) \) be defined by the unique solution to

\[
E[u(V_1 - v(x, y))|X_1 = x, Y_1 = y] = u(0).
\]

We restrict our attention to the class of maximal attentive common value auctions that can be defined simply by the requirement

\[
y \leq x, \quad y' \leq x \quad \text{imply} \quad v(x, y) = v(x, y').
\]

This implicit restriction upon \( h(X|V) \) is ensured by the requirement that \( x_m := \max \{x_1, \ldots, x_m\} \) is a sufficient statistic for the observations \( (x_1, \ldots, x_m) \) drawn from \( h(X|V) \). This implies that knowing \( x_m \) and the number of observations \( m \) leaves an observer as well placed to estimate \( V \) as if he observed the entire vector from which \( x_m \) is maximal (Blackwell (1952)).

A specific example of an auction in the maximal attentive class satisfying (3) can be constructed as follows. Let \( g(v) \) be uniform on \([0, v]\) and let \( h(X|V = v) \) be uniform on \([0, v]\). Then the estimates of \( V \) from the sets of observations \((25, 24, 24.4)\) and \((25, 3, 2)\) are identical.\(^6\)

This requirement (3) may be a plausible idealization of such auction markets as offshore oil leases in previously unleased areas, where an estimate from seismic soundings is the informative signal.\(^7\) Constructions like the example can be induced in the laboratory, and contrasted with constructions which fail to satisfy (3).

Without (3), Matthews (1977) and Milgrom (1981) show that, for the function \( b^*(x) = v(x, x) \), \((b^*, \ldots, b^*)\) is a Nash equilibrium. Employing (3), their argument can
be strengthened to dominance solvability: any bid below \( \nu(x, x) \) is dominated, and if no opponent bids below \( \nu(x, x) \), this allows all higher bids to be dominated.

**Theorem 1.** For the maximal attentive class of auctions, bidding \( b^*(x) = \nu(x, x) \) dominates any lower alternative \( b_i(x) \leq \nu(x, x) \) for any opposition \( (b_i)_{i=1}^n \).

**Proof.** Let bidder 1 observe \( X_1 = x_0 \), and consider his choice between \( b^*(x_0) \) and any \( b_i < b^*(x_0) \). If \( p < b_i \), both bids win at price \( p \). If \( p > b^*(x_0) \), neither wins. Thus, the choice between them is decided by the event

\[ \Theta := \{ b^*(x_0) \geq p \geq b_i \}. \]

If \( \Pr(\Theta) = 0 \), \( b^*(x_0) \) weakly dominates \( b_i \). If \( \Pr(\Theta) > 0 \), \( b_i \) earns expected utility \( 0 \), while \( b^*(x_0) \) earns

\[
E[u(V-p)|X_1=x_0, \Theta] \geq E[u(V-\nu(x_0, x_0))|X_1=x_0, \Theta] \\
\geq E[u(V-\nu(x_0, x_0))|X_1=x_0, Y_1=x] \\
= E[u(V-\nu(x_0, x_0))|X_1=x_0, Y_1=Y_1=0],
\]

where \( x \) is the lowest possible signal, the inequalities result from the event \( \Theta \) (the second is the most pessimistic possibility given \( \Theta \)), and the equalities from (3) and (2).

The second step in the argument depends upon bidders recognizing the consequences of Theorem 1.

**Theorem 2.** For the maximal attentive attentive class of auctions, if \( b_i(x) \geq \nu(x, x) \), for \( i = 2, \ldots, n \), then bidding \( b^*(x) = \nu(x, x) \) dominates any alternative \( b_i(x) \).

**Proof.** Given Theorem 1, it suffices to compare, for \( X_1 = x_0 \), \( b^*(x_0) \) and an arbitrary \( b_i > b^*(x_0) \). This choice is decided by the event

\[ \Theta' := \{ b_1 \geq p \geq b^*(x_0) \}, \]

and \( b^*(x_0) \) weakly dominates unless \( \Pr(\Theta') > 0 \). Let \( \bar{x} := b^*^{-1}(p) \), which is well-defined because \( b^* \) is increasing, even with (3). (Thus, \( p = b^*(\bar{x}) = \nu(\bar{x}, \bar{x}) \).) In event \( \Theta' \), \( \bar{x} \geq x_0 \) and since \( b_j(x) \geq \nu(x, x) \); \( j = 1 \), any bidder \( j \) observing \( X_j = \bar{x} \) would have bid more than \( p \).

Combining, bidding \( b^*(x_0) \) earns zero in event \( \Theta' \), while \( b_i \) earns

\[
E[u(V-p)|X_1=x_0, \Theta'] = E[u(V-\nu(\bar{x}, \bar{x}))|X_1=x_0, \Theta'] \\
\geq E[u(V-\nu(\bar{x}, \bar{x}))|X_1=x_0, Y_1=x] \\
\geq E[u(V-\nu(\bar{x}, \bar{x}))|X_1=Y_1=x]=0,
\]

with inequalities due to the most optimistic possibility given \( \Theta' \), and affiliation.

These straightforward dominance solvable auctions offer an opportunity to contrast (e.g. in the laboratory) the predictive power of Nash equilibrium and dominance criteria.

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NOTES
1. In a second-price auction, the highest bidder obtains the item at a price equal to the second-highest bid. A dominant strategy is a strategic choice that does at least as well as any alternative for every pattern of the behaviour of other players.
2. Moulin (1979) introduces the concept in the context of voting games. It is generalized in Moulin (1982), and a more elementary presentation is in Moulin (1982), Chapter 2.
3. The model is the General Symmetric Model of Milgrom and Weber (1982). Sections 3, 4 and 8 in their paper discuss assumptions and motivations in more detail. We ignore their vector S of nonparticipant appraisals.
4. Let z and z' be points in $R^{n-1}$, and $z \times z'$ denote the component-by-component maximum of z and z', with $z \times z'$ the component-wise minimum. Then affiliation requires

$$f(z \times z')/f(z) \geq f(z')/f(z')$$

This equation is satisfied with equality for independent variables. See Milgrom and Weber (1982), pp. 1098-1100 and Appendix, for more detail.
5. Nondecreasingness is implied by affiliation.
6. In the case where, given u, the X, are conditionally independent, the maximal attentive class generalizes this example to allow $X_i = (Z_i, \theta)$, $\theta > 0$, where $Z_i$ is uniform on $[k_i, k_i + k_i]$, constants known. This is the model analyzed in Matthews (1984), with $\theta$ endogenous. Note signals need not be lower bounds for $V$.
7. This example admitted cannot accommodate a probability mass at $V = 0$. In some cases, auctions where bidders are buying for storage or renovation, followed by resale in the same forum, may admit to maximal attentiveness as a plausible idealization.
8. If $b_i = p$, 1 wins with probability $\Pi \equiv 1/2$, the first inequality below is strict, and so 1 prefers to bid $b^*(x_0)$ and win with probability $\Pi = 1$.
9. If $b^*(x_0) = p$, then 1 wins with probability $\Pi \equiv 1/2$, and no inequality to follow is strict.

REFERENCES