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Auction form preferences of risk-averse bid takers

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We analyze the preferences of a risk-averse seller over the class of "standard" auctions with symmetric and risk-neutral bidders. Assuming that buyers' private signals are independently distributed, we find that a sealed-bid first-price auction with an appropriately set reserve price is preferred by all risk-averse sellers to any other standard auction. In first- and second-price auctions, the more risk averse a seller, the lower the seller's optimal reserve price. Given two first-price auctions with reserve prices and entry fees such that both have the same screening level, all risk-averse sellers prefer the auction with the lower entry fee.

1. Introduction

■ Much of the theoretical literature on auctions is concerned with a comparison of revenue from different auction forms. A seller choosing between auction forms presumably prefers the form that generates the highest equilibrium expected revenue. Such a research agenda implicitly assumes, however, that the seller is risk neutral. This article generalizes these analyses by considering preferences of risk-averse sellers over standard auction forms when selling to risk-neutral bidders. This is a natural step in trying to model such markets as manuscript auctions, where publishers have diversification opportunities but an individual author does not. Initial public offerings by small firms, where the current owner's equity and the capital inflow are at risk, but equity variability of the security brokers bidding to underwrite is negligibly at stake, are also examples.

When bidders' private information is independently distributed (as in the independent-private-values environment), all of the standard auction forms generate the same

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expected revenue.[1] This result is commonly referred to as "revenue equivalence." In addition, with an appropriately set reserve price, all of the standard auction forms maximize expected revenue. In contrast to the revenue equivalence result, a risk-averse seller is not indifferent between all standard auction forms.

Risk aversion creates a preference for a first-price sealed-bid auction with an optimally set reserve price to all other standard auction forms when buyers' private information is independently distributed. Intuitively, this preference arises because, conditioned on the highest signal, all standard auctions yield the same expected revenue, but conditioned on the highest signal, revenue in a first-price auction is nonstochastic; under the same conditioning, revenue in other auctions retains its randomness. As we show, this extra randomness makes all other standard auction forms inferior to a first-price auction from the perspective of risk. Note that it would appear to be possible to make a similar argument for second-price auctions relative to first-price: conditional on the second-highest bidder's type, second-price auction revenue is nonstochastic, while first-price auction revenue would still be random. This argument fails, however, since conditional on the second-highest signal, the expected revenues from first-price and second-price auctions are not equal.

A risk-averse seller also prefers second-price to English auctions when the signals are independently distributed. The intuition underlying this result is similar in that, conditioned on the highest two types, the revenue from a second-price auction is nonstochastic while English auction revenue retains some randomness.

In first-price auctions, an increase in the reserve price increases the equilibrium bids of participating bidders but also increases the probability that no bidders will participate in the auction. While not mean preserving, an increase in the reserve price shifts probability weight to the upper and lower tails of the revenue distribution. We show that a seller's optimal reserve price is decreasing in the degree of the seller's risk aversion. The same comparative static result is shown for second-price auctions when the winning bidder's expected value is independent of the private information of rivals.

Given two first-price auctions with different reserve prices and entry fees set so as to yield the same expected revenue from independently informed bidders, a risk-averse seller prefers the auction with the lower entry fee. Entry fees introduce an additional randomness to the revenue generated by an auction, since the number of bidders paying the fee is random. When the entry fee is higher, more of the revenue from the auction is generated from that source, increasing the risk in overall revenue.

Auction models often make predictions about unobservable variables, such as bidders' asset values. In contrast, our results relate to the price variability generated by different auction forms. As is shown in Theorem 1, all risk-averse sellers prefer a first-price auction to any other standard auction; thus, price variability in first-price auctions must be less than the variability in other standard auctions. An empirical test of equilibrium behavior could therefore be constructed on the basis of these results, using price data from auctions of different forms.

Symmetrically risk-averse buyers in the private-values environment have been extensively studied. A first-price auction is shown to generate higher expected revenue than second-price and English auctions when bidders are risk averse.[2] Maskin and Riley

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[1] To be precise, any auction form for which the highest valuer buys the asset for certain, and a bidder with the lowest possible private information attains zero expected profit, yields a revenue equal in expectation to the second-highest evaluation; see Myerson (1981).

[2] See Harris and Raviv (1981), Holt (1980), Matthews (1980), and Riley and Samuelson (1981); Matthews (1987) generalizes, by also considering buyers' preferences over different auction forms. Smith and Levin (1996) generalize these results by allowing for endogenous participation in the auction. In these articles the assumption that buyers know for certain the auctioned asset's value to them is peculiar, as it leaves the only risk that of losing the auction.

(1984) find in a more general case that when bidders are risk averse, the expected-revenue-maximizing selling mechanism is so complex that it hardly resembles an auction.

Few articles consider the preferences of risk-averse sellers over different auction forms. Vickrey's (1961) classic article calculates the variance of prices in first-price and English auctions with uniformly distributed private values and risk-neutral bidders.[3] Matthews (1980) and Maskin and Riley (1984) consider the preferences of a risk-averse seller over first-price and second-price auctions in the private-values environment when the buyers are symmetrically risk averse (or risk neutral). Our analysis of first-price auctions differs from theirs in that we show that risk-averse sellers prefer first-price auctions not only to second-price auctions but to any standard auction. Also, we allow bidders to be uncertain of asset value, incorporating some common-value elements. In order to focus on seller risk aversion, we assume bidders are risk neutral.

For concreteness, bid takers are referred to as sellers and bidders as buyers. However, a complete analogue of all the results can readily be extended to markets in which a risk-averse buyer seeks to buy via auction from a collection of risk-neutral sellers. Examples would include cases in which small firms seek to have relatively specialized plant or equipment constructed in a one-time contract, with the cost or the consequences of substandard performance running to a large fraction of equity.

2. The model

■ We analyze the "general symmetric model" of affiliated-values auctions in Milgrom and Weber (1982); a brief outline is provided here, with readers encouraged to consult Milgrom and Weber's presentation for motivation of the assumptions. Our analysis does not require a reevaluation of equilibrium bidding behavior: once the seller has committed to an auction form, the incentives faced by the bidders are unaffected by the risk preferences of the seller.

A (strictly) risk-averse seller is any seller whose preferences over different auction forms are described by her expected utility of the resulting revenue, where the utility function is bounded, increasing, (strictly) concave, and (for simplicity) differentiable (Rothschild and Stiglitz, 1970). We use U to denote such utility functions.

The seller wishes to auction off a single indivisible asset. Let $N = \{1, \ldots, n\}$ denote the set of risk-neutral bidders. Prior to the auction, each bidder i privately observes a signal, $X_i \in [\underline{x}, \bar{x}]$, containing some information about his value for the asset. Let $X = (X_1, \ldots, X_n)$. Bidder i's value for the object at auction is denoted $V_i = u(X_i, X_{-i}, S)$, where $X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, $S = (S_1, \ldots, S_m)$ is a vector of other relevant information, and the support of V_i is bounded. The function u is positive and differentiable. Furthermore, we assume that u is increasing in X_i and symmetric and nondecreasing in X_{-i}. Let v_0 denote the seller's deterministic asset value. Let $Y_1, \ldots, Y_{n-1}$ denote a reordering of $X_2, \ldots, X_n$ such that $Y_1 \geq Y_2 \geq \ldots \geq Y_{n-1}$. (Thus, $Y_1 = \max\{X_2, \ldots, X_n\}$.) We assume that (S, X) are affiliated and that the joint density, f, of (S, X) is symmetric and differentiable in X.[4] Define the function v as $v(x, y) = E[u(x, X_{-1}, S) \mid X_1 = x, Y_1 = y]$. Affiliation and the monotonicity of u imply that v is increasing in its first argument and nondecreasing in its second argument; the

[3] Vickrey (1961) calculates the variances correctly but misreports their relative quantities.

[4] Random variables jointly distributed by density f are affiliated if, for all z and z',

$$f(z \vee z')f(z \wedge z') \geq f(z)f(z'),$$

where $z \vee z' = (\max\{z_1, z_1'\}, \ldots, \max\{z_n, z_n'\})$ and $z \wedge z' = (\min\{z_1, z_1'\}, \ldots, \min\{z_n, z_n'\})$. The results that we use relating to affiliated random variables are all presented by Milgrom and Weber (1982).

differentiability of v follows from the differentiability of u and f. A bidding strategy is a function from the support of X_i to $\mathbf{R}_+$. Throughout this article we restrict attention to an equilibrium where all bidders follow a common pure strategy.

The affiliated-values environment has the benefit of containing many of the environments found in the theoretical auction literature as special cases. The common-value environment is recovered by adding the assumption that $E[V_i|X] = E[V_j|X]$ for all $i, j \in N$. In the private-values environment u depends only on X_i. Much of the literature on auctions deals with the independent-private-values environment, which is recovered by the further assumption that the private signals are independently and identically distributed. Three of the results in this article, Theorems 1, 2, and 4, apply in the special case of the affiliated-values environment where the signals $X_1, \ldots, X_n$ are independently distributed. This assumption generalizes the independent-private-values environment, since independence of the signals is necessary but not sufficient to yield that environment. In the common-value environment, it is standard to interpret the private signals as estimates of the common value. Such an interpretation would lead one to conclude that the signals should be correlated. Technically speaking, however, independent signals are consistent with the common-value environment. An example of such a common-value environment occurs when the common value in an auction is the mean of the bidders' independent private signals (see Albers and Harstad, 1991). Thus, the results of Theorems 1, 2, and 4 hold for common-value environments with independent signals.

We only consider auctions where participation is voluntary and nonparticipating bidders receive a zero payoff. (Nonparticipating bidders have no chance of receiving the asset and pay nothing to the seller.) Hence, if a bidder's expected payoff from participating in the auction is negative, then he will choose not to participate. Auction rules sometimes require a minimum bid (reserve price) or an entry fee from participating bidders. These rules could result in a negative expected payoff for some bidders if they choose to participate (i.e., pay the entry fee, if any, and bid). When a common bidding strategy is followed and bidders who are indifferent between participating and not participating choose to participate, then the monotonicity of v implies that there is a screening level, $x^s \in [\underline{x}, \overline{x}]$ such that in equilibrium any and only bidders with signals below x^s choose not to participate in the auction. We limit our attention to auction rules for which bidders choose to participate with positive probability (i.e., $x^s < \overline{x}$).

In Theorems 1, 2, and 3 we consider the revenue generated by standard auctions from the perspective of a risk-averse seller. A standard auction is a bidding scheme where (in equilibrium) the asset is always allocated to the bidder with the highest private signal as long as that signal is above the screening level, the winning bidder makes a nonnegative payment, the nonwinning bidders pay zero, the auction rules are anonymous, and there is a common equilibrium strategy.[5] In general, the price paid by the winning bidder may be random or might be a function of any of the bids received. Most of the commonly studied auction forms (first-price, second-price, and the stylized English) can be modelled as standard auctions.

The following lemma and corollary are slight variations on results commonly found in the auction literature (see Myerson, 1981). We use these particular versions of the results as lemmas in proofs presented later.

Lemma 1 (expected payment equivalence). Consider auction forms A and B with the same screening level. Suppose that $X_1, \ldots, X_n$ are independent and that in both forms

[5] Our definition generalizes Milgrom's (1987) definition of a "standard" auction in that it includes auctions that do not always award the object to a buyer, as in the case when no bids are received over a reserve price. However, our definition is a special case of the classes of auctions analyzed by Myerson (1981) and Riley and Samuelson (1981).

the asset is always allocated to the bidder with the highest private signal as long as that signal is above the screening level. Then conditional on his private signal, a bidder's expected payment is equal in both auction forms.

Proof. Let $\pi_\xi(z, x)$ denote bidder 1's expected payoff in auction $\xi \in \{A, B\}$ when bidder 1 has private signal x, all the other bidders follow their common equilibrium strategy, and bidder 1 follows the equilibrium strategy of a bidder with signal z. For $z \geq x^s$, $\pi_\xi(z, x) = \phi(z, x) - p_\xi(z)$, for $\xi \in \{A, B\}$, where $\phi(z, x) = \int_{\underline{x}}^{z} v(x, y) f_{Y_1}(y)\, dy$ and $p_\xi(z)$ denotes bidder 1's expected payment when he follows the equilibrium strategy of a bidder with private signal z in auction form ξ. The independence of $X_1, \ldots, X_n$ implies that a bidder's expected payment depends on z but not on x. Incentive compatibility implies $\phi(x, x) - p_\xi(x) \geq \phi(z, x) - p_\xi(z)$ and $\phi(z, z) - p_\xi(z) \geq \phi(x, z) - p_\xi(x)$. For $x \gtreqless z$, these conditions imply

$$\frac{\phi(x, x) - \phi(z, x)}{x - z} \gtreqless \frac{p_\xi(x) - p_\xi(z)}{x - z} \gtreqless \frac{\phi(x, z) - \phi(z, z)}{x - z}.$$

Taking the limit of the expression above as z approaches x results in $v(x, x) f_{Y_1}(x) = p_\xi'(x)$. π_ξ must also satisfy $\pi_\xi(x^s, x^s) = 0$, since if $\pi_\xi(x^s, x^s) < 0$, then a bidder with signal x^s would prefer not to participate, and if $\pi_\xi(x^s, x^s) > 0$, then some bidders with signals below x^s would have an incentive to participate, violating the definition of x^s. Using these conditions to solve for p_ξ yields $p_\xi(x) = \int_{x^s}^{x} v(y, y) f_{Y_1}(y)\, dy + \phi(x^s, x^s)$ for $x \geq x^s$ and $\xi \in \{A, B\}$. We also know that $p_\xi(x) = 0$ for $x < x^s$, since nonparticipating bidders pay nothing. Therefore, $p_A(x) = p_B(x)$ for all x. *Q.E.D.*

Corollary 1 (expected revenue equivalence). Let R_A and R_B denote the revenue generated by the two auctions described in Lemma 1.

(i) $E[R_A] = E[R_B]$.
(ii) Further suppose that A and B are standard auctions. Then

$$E[R_A | X_1 = x, Y_1 < x] = E[R_B | X_1 = x, Y_1 < x].$$

Proof. (i) $E[R_A] = \sum_{i=1}^{n} E[p_A(X_i)] = \sum_{i=1}^{n} E[p_B(X_i)] = E[R_B]$.

(ii) In a standard auction, the only bidder to make a payment to the seller is the winning bidder (who is also the bidder with the highest private signal). Let $\tilde{p}_\xi(x)$ be the expected payment of a bidder with signal x in auction ξ conditional on having the highest signal. Hence, $p_\xi(x) = \tilde{p}_\xi(x) F_{Y_1}(x)$. Therefore,

$$E[R_A | X_1 = x, Y_1 < x] = \tilde{p}_A(x) = \tilde{p}_B(x) = E[R_B | X_1 = x, Y_1 < x],$$

since all of the seller's revenue derives from the winning bidder's payment. *Q.E.D.*

As is common, our analysis focuses on first-price, second-price, and English auctions, all of which are standard auctions when there is no entry fee. In a first-price auction, bids are submitted simultaneously; if the reserve price is met, then the highest bidder obtains the asset for the amount of his bid. A second-price auction operates similarly, except that the price is the greater of the second-highest bid and the reserve price. In an English auction, the price is raised orally until there is only one bidder left active. This general theme can take a number of different forms.[6] We analyze the

[6] Rothkopf and Harstad (1991) show that these different forms of the English auction have different implications for the revenue of the seller.

stylized version of the English auction considered by Milgrom and Weber (1982), where the price rises continuously and bidders' exits are public and irrevocable.

3. The seller's preferences over standard auction forms

■ For the special case of the affiliated-values environment where the signals X are independently distributed, a risk-averse seller's preference over standard auctions is definitive.

Theorem 1. If $X_1, \ldots, X_n$ are independent, then (i) all risk-averse sellers prefer a first-price auction to any other standard auction with the same screening level, and (ii) any strictly risk-averse seller strictly prefers a first-price auction to any standard auction with the same screening level and where in the standard auction the distribution of the payment by the winning bidder conditioned on his private signal is not degenerate.

Proof. (i) Let R_F and R_A denote the revenue from a first-price auction and from some other standard auction form, both with screening level x^s. Let 1_θ be the indicator function taking the value one in event θ, zero otherwise.

$$\begin{aligned} E[U(R_A + v_0 \cdot 1_{\{x<x^s\}}) | X_1 = x, Y_1 < x] &\leq U(E[R_A + v_0 \cdot 1_{\{x<x^s\}} | X_1 = x, Y_1 < x]) \\ &= U(E[R_F + v_0 \cdot 1_{\{x<x^s\}} | X_1 = x, Y_1 < x]) \\ &= E[U(R_F + v_0 \cdot 1_{\{x<x^s\}}) | X_1 = x, Y_1 < x]. \end{aligned}$$

The first line in the chain of inequalities follows from Jensen's inequality, the second line from part (ii) of Corollary 1, and the final equality from R_F being conditionally nonstochastic. Define the random variable $Z = \max\{X_1, \ldots, X_n\}$. The desired unconditional preference, that $E[U(R_A + v_0 \cdot 1_{\{Z<x^s\}})] \leq E[U(R_F + v_0 \cdot 1_{\{Z<x^s\}})]$, follows when x is replaced with X_1 in the inequality above and the unconditional expectation is taken.[7]

(ii) The assumptions imply that (a) U is strictly concave and (b) conditioned on $X_1 = x$ and $Y_1 < x$, R_A is not degenerate conditioned on $X_1 = x$ and $Y_1 < x$. Together, (a) and (b) suffice for a strict version of Jensen's inequality, completing the proof of part (ii). *Q.E.D.*

An immediate corollary of Theorem 1 is that a first-price auction is strictly preferred to both second-price and English auctions by strictly risk-averse sellers, since the resulting prices in both second-price and English auctions depend on the bids of the nonwinning buyers. The results of Theorem 1 conform with Vickrey (1961), Maskin and Riley (1984), and Matthews (1980).[8]

Theorem 2. If $X_1, \ldots, X_n$ are independent, then all risk-averse sellers prefer a second-price to an English auction with the same screening level.

Proof. Consider an English auction with an arbitrary reserve price r. Let x^s denote the common screening level resulting from the common reserve price r.

Define the function $\bar{v}$ as

$$\bar{v}(x, y_1, \ldots, y_{n-1}) = E[V_1 | X_1 = x, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}].$$

Let $R_S(x, y)$ and $R_E(x, y)$ denote the expected revenue from a second-price auction and

[7] We thank an anonymous referee for pointing out this relatively straightforward proof for the result.

[8] Experimental evidence presented by Kagel, Harstad, and Levin (1987) and Coppinger, Smith, and Titus (1980) also shows higher variance for second-price auctions than for first-price auctions despite variance-reducing constraints in Coppinger, Smith, and Titus (1980) preventing bids above values in second-price auctions.

an English auction respectively with reserve price r, conditional on the highest signal received by any buyer being x and the second-highest signal observed being y. Using Milgrom and Weber's (1982) derivations of the equilibrium strategies, R_S and R_E can be written as

$$R_S(x, y) = \begin{cases} 0 & \text{for } x < x^s \\ r & \text{for } y < x^s \le x \\ v(y, y) & \text{for } y \ge x^s \end{cases}$$

$$R_E(x, y) = \begin{cases} 0 & \text{for } x < x^s \\ r & \text{for } y < x^s \le x \\ \overline{v}(y, y, Y_2, \ldots, Y_{n-1}) & \text{for } y \ge x^s. \end{cases}$$

Notice that the independence of X implies

$$E[R_S(x, y) | X_1 = x, Y_1 = y] = E[R_E(x, y) | X_1 = x, Y_1 = y].$$

The remainder of the argument closely parallels the proof of Theorem 1.

$$\begin{aligned} E[U(R_E + v_0 \cdot 1_{\{x<x^s\}}) | X_1 = x, Y_1 = y] &\le U(E[R_E + v_0 \cdot 1_{\{x<x^s\}} | X_1 = x, Y_1 = y]) \\ &= U(E[R_S + v_0 \cdot 1_{\{x<x^s\}} | X_1 = x, Y_1 = y]) \\ &= E[U(R_S + v_0 \cdot 1_{\{x<x^s\}}) | X_1 = x, Y_1 = y]. \end{aligned}$$

The conclusion follows when x, y are replaced with X_1, Y_1 and the expectation conditional on $X_1 > Y_1$ is taken. *Q.E.D.*

Taken together, Theorems 1 and 2 imply that all risk-averse sellers prefer first-price to second-price auctions and second-price to English auctions when the signals $X_1, \ldots, X_n$ are independently distributed. However, Milgrom and Weber (1982) find exactly the reverse preference structure, English to second-price to first-price auctions, for a risk-neutral seller in the general affiliated-values environment. This reverse preference structure will carry over to a slightly risk-averse seller when signals are strictly affiliated, thus Theorems 1 and 2 cannot hold without the assumption that the signals are independent.

4. Risk aversion and reserve-price choice

■ Besides affecting a seller's preferences over auction forms, the risk preferences of a seller will affect the choice of a reserve price. A seller with utility function U_M is said to be more risk averse than a seller with utility function U_L, if $U_M(U_L^{-1}(t))$ is a strictly concave function of t (Pratt, 1964). Let $b(\cdot | r)$ denote the equilibrium bidding strategy in a first-price auction with a reserve price of r. In a first-price or second-price auction with reserve price r, the screening level, $x^s(r)$, is defined implicitly by $\int_{\underline{x}}^{x^s} [v(x^s, y) - r] f_{Y_1}(y | x^s)\, dy = 0$. For a given reserve price, the same set of bidders would participate in a first-price auction as would participate in a second-price auction (see Milgrom and Weber, 1982). The following theorem relates a seller's risk preferences to her choice of a reserve price in first-price and second-price auctions.[9] To avoid a corner solution for the optimal reserve price, we assume that v_0 satisfies

[9] For a first-price auction in the independent private-values environment with the monotone hazard rate assumption, Matthews (1980) shows that a risk-neutral seller would set a higher reserve price than a risk-averse seller would.

$$E[V_1 | X_1 = \bar{x}, Y_1 < \bar{x}] > v_0 > v(\underline{x}, \underline{x}). \tag{1}$$

Theorem 3. Suppose that v_0 satisfies (1) and a seller with utility function U_M is more risk averse than a seller with utility function U_L.

(i) Then in a first-price auction, a seller with utility function U_L will set a reserve price that is higher than the reserve price set by a seller with utility function U_M.

(ii) Further suppose that $v(x, y)$ is constant in y, for $y \leq x$. Then in a second-price auction, a seller with utility function U_L will set a reserve price that is higher than the reserve price set by a seller with utility function U_M.

The proof of this theorem makes use of the following two lemmas.

Lemma 2. In a first-price auction with reserve price r and equilibrium bid function $b(x|r)$, $\partial b(x|r)/\partial r > 0$, for all x above the screening level and all r such that $E[V_1 | X_1 = \bar{x}, Y_1 < \bar{x}] > r > v(\underline{x}, \underline{x})$.

Proof. See the Appendix.

Lemma 3. Consider two functions g_i: $\mathbf{R} \rightarrow \mathbf{R}$ for $i = 1, 2$ that are differentiable on a convex set $S \subset \mathbf{R}$ and where $\operatorname{argmax}\{g_i(x)\} \subset S$, for both $i = 1, 2$ and $g_1'(x) > g_2'(x)$, for all $x \in S$. Then for any $x_1^* \in \operatorname{argmax}\{g_1(x)\}$ and $x_2^* \in \operatorname{argmax}\{g_2(x)\}$, $x_1^* > x_2^*$.[10]

Proof. See the Appendix.

Proof of Theorem 3. (i) A seller's expected utility from a first-price auction with reserve price r and utility function U_Φ for $\Phi = M, L$ is

$$H_\Phi(r) = \int_{x^s(r)}^{\bar{x}} U_\Phi(b(x|r))\tilde{f}(x)\, dx + U_\Phi(v_0)\tilde{F}(x^s(r)), \tag{2}$$

where $\tilde{F}$ and $\tilde{f}$ are the distribution and density functions for the highest signal and $x^s(r)$ is the screening level. It is straightforward to check that $b(x^s(r)|r) = r$ (see the expression for b in the proof of Lemma 2). Our assumptions imply that $x^{s\prime}(r) > 0$, for r such that $x^s(r) \in (\underline{x}, \bar{x})$ (see the proof of Lemma 2, first paragraph).

Let $S = (v_0, \bar{r})$, where $\bar{r} = E[V_1 | X_1 = \bar{x}, Y_1 < \bar{x}]$. (Note that $x^s(\bar{r}) = \bar{x}$.) For any increasing utility function U_Φ, $\operatorname{argmax}_r\{H_\Phi(r)\} \subset S$. To see this, note the following. The facts that $b(x|r)$ is increasing in r and x and $b(x^s(v_0)|v_0) = v_0$ imply that for $r \geq \bar{r}$, $H_\Phi(r) = U_\Phi(v_0) \leq H_\Phi(v_0)$. Notice that for $r \leq v_0$,

$$H_\Phi'(r) = [U_\Phi(v_0) - U_\Phi(r)]\tilde{f}(x^s)x^{s\prime}(r) + \int_{x^s}^{\bar{x}} U_\Phi'(b(x|r))\frac{\partial b(x|r)}{\partial r}\tilde{f}(x)\, dx \geq 0,$$

since U_Φ, x^s, and b are increasing functions and since the screening level and bid function are the same for any reserve price less than or equal to $v(\underline{x}, \underline{x})$. The inequality is strict for $r = v_0$. Therefore, $\operatorname{argmax}_r\{H_\Phi(r)\} \subset S$. By Lemma 3, to complete the proof of part (i) it is sufficient to fix an arbitrary $r^\circ \in S$ and to show $H_L'(r^\circ) - H_M'(r^\circ) > 0$.

Without loss of generality, the risk preferences in U_M and U_L can be maintained while normalizing each at two points. It is convenient to set two fixed points, $U_M(v_0) = U_L(v_0) = v_0$ and $U_M(r^\circ) = U_L(r^\circ) = r^\circ$. Define $h(\cdot) = U_M(U_L^{-1}(\cdot))$; h is strictly concave. Hence, by construction, the graph of h passes through (v_0, v_0) and (r°, r°), and $h'(t) < 1$ for $t \geq r^\circ$. Thus, $U_M'(\xi) < U_L'(\xi)$ for $\xi \geq r^\circ$, since $U_M'(\xi) = h'(U_L(\xi))U_L'(\xi)$. The fact that $x^s \in (\underline{x}, \bar{x})$ follows from the definition of x^s and the fact that $r^\circ \in S$. Hence,

[10] This result was suggested by an anonymous referee.

$$H'_L(r^\circ) - H'_M(r^\circ) = \int_{x^s(r^\circ)}^{\bar{x}} [U'_L(b(x|r^\circ)) - U'_M(b(x|r^\circ))] \frac{\partial b(x|r^\circ)}{\partial r} \tilde{f}(x)\, dx > 0,$$

since $b(x|r^\circ) \geq r^\circ$, for all $x \geq x^s(r^\circ)$.

(ii) The proof of the theorem for a second-price auction is similar. For this part of the proof let $H_\Phi(r)$ denote a seller's expected utility from a second-price auction with reserve price r and utility function U_Φ for $\Phi = M, L$. That is,

$$H_\Phi(r) = \int_{x^s(r)}^{\bar{x}} U_\Phi(v(y, y)) \int_y^{\bar{x}} \hat{f}(z, y)\, dz\, dy + U_\Phi(r) \int_{x^s(r)}^{\bar{x}} \int_{\underline{x}}^{x^s(r)} \hat{f}(z, y)\, dy\, dz$$
$$+ U_\Phi(v_0) \int_{\underline{x}}^{x^s(r)} \int_{\underline{x}}^{x^s(r)} \hat{f}(z, y)\, dy\, dz,$$

where $\hat{f}$ is the joint distribution of the highest and second-highest private signals.

Let S retain its definition from the proof of part (i). Under the assumption that v is constant in its second argument the implicit definition for the screen level can be written $v(x^s(r), x^s(r)) = r$. For $r < v_0$,

$$H_\Phi(v_0) - H_\Phi(r) = \int_{x^s(r)}^{x^s(v_0)} \int_y^{\bar{x}} [U_\Phi(v_0) - U_\Phi(v(y, y))] \hat{f}(z, y)\, dz\, dy$$
$$+ [U_\Phi(v_0) - U_\Phi(r)] \int_{x^s(r)}^{\bar{x}} \int_{\underline{x}}^{x^s(r)} \hat{f}(z, y)\, dy\, dz > 0.$$

The inequality follows from the fact that for all $y < x^s(v_0)$, $v_0 > v(y, y)$. To see this, notice that when v is constant in its second argument, $r = v(x^s(r), x^s(r))$ (by implicit definition of the screening level) and, hence, $v_0 = v(x^s(v_0), x^s(v_0)) > v(y, y)$. A seller will never choose a reserve price greater than $\bar{r}$, since $H_\Phi(r) < H_\Phi(v_0)$ for all $r \geq \bar{r}$. It is straightforward to show that $H'_\Phi(v_0) > 0$. Thus, $\text{argmax}_r\{H_\Phi(r)\} \subset S$ for both $\Phi = M$ and $\Phi = L$, and by Lemma 3, it is sufficient to fix an arbitrary $r^\circ \in S$ and to show $H'_L(r^\circ) - H'_M(r^\circ) > 0$.

Again letting U_M and U_L have the fixed points at v_0 and r° implies $U'_M(r^\circ) < U'_L(r^\circ)$, so

$$H'_L(r^\circ) - H'_M(r^\circ) = [U'_L(r^\circ) - U'_M(r^\circ)] \int_{x^s(r^\circ)}^{\bar{x}} \int_{\underline{x}}^{x^s(r^\circ)} \hat{f}(z, y)\, dy\, dz > 0.$$

Q.E.D.

The comparative static result presented in Theorem 3 holds for the generalized affiliated-values environment in the case of first-price auctions, but we use a rather strong assumption on valuations to prove the result for second-price auctions.[11] As is

[11] One might suspect that this assumption implies the private-values model. However, that is not the case. "Maximal attentive" common-value environments, as defined by Harstad and Levin (1985), satisfy the assumption that $v(x, y)$ is degenerate in y when $y \leq x$. This class of environments is defined by the characteristic that the highest signal is a sufficient statistic for all of the bidders' signals. Harstad and Levin (1985) show that under this assumption second-price auctions are dominance solvable.

demonstrated in Example A1 (in the Appendix), without the assumption that v is invariant with respect to its second argument, the result does not hold. In that example we show that when $v(x, y) = x + y$, it is possible that in a second-price auction a risk-averse seller would want to set a higher reserve price than a risk-neutral seller would. When v is strictly increasing in its second argument, $v(x^s, x^s) > r$, and when v is constant in its second argument, $v(x^s, x^s) = r$. Our proof of Theorem 3, part (ii) requires that $v(x^s, x^s) = r$, and Example A1 only works because in it $v(x^s, x^s) > r$.

A natural intuition for the result presented in Theorem 3 is that the disutility of failing to sell the asset when values fall below the screening level looms larger the more risk averse the seller is. However, the result (and hence this intuition) does not extend to second-price auctions in the general affiliated-values case. The result fails in cases where there are no bids submitted in a neighborhood of the reserve price. (In Example A1, when the reserve price is .6, the lowest submitted bid is in fact .8.) The additional condition used in part (ii) of Theorem 3 ensures that bids are submitted with positive probability in any neighborhood of the reserve price.

5. Auctions with entry fees

■ Thus far, we have considered auction forms in which losing bidders pay nothing. Some auction forms, however, involve payments by losers, as in a first-price auction with an entry fee (a fee to permit bid submission). As is standard (see, e.g., Milgrom and Weber, 1982), we assume that decisions to pay the entry fee are made simultaneously, and a bidder is not informed of how many others have paid the fee before bidding. A bidder decides whether or not to pay the entry fee and, if paying, submits a bid without knowing how many of the $n - 1$ other bidders will pay.

Theorem 4. Consider two first-price auctions with reserve price and entry fee combinations (r, e) and (r', e') such that $e < e'$ and such that the two auctions have the same screening level. If $X_1, \ldots, X_n$ are independent, then all risk-averse sellers prefer the auction with the lower entry fee.

Proof. Let $b_{(r,e)}$ and $b_{(r',e')}$ denote the equilibrium bidding strategies in the auctions with (r, e) and (r', e'). Let x^s denote the common screening level. Lemma 1 implies

$$b_{(r,e)}(x)F_{Y_1}(x) + e = b_{(r',e')}(x)F_{Y_1}(x) + e' \tag{3}$$

for all $x \geq x^s$. Define Ω as the number of bidders paying entry fees in auctions (r, e) and (r', e'). $\Omega = 1_{\{X_1 \geq x^s\}} + \ldots + 1_{\{X_n \geq x^s\}}$. Notice that Ω is the same for both auctions, since both have the same screening level. Therefore, the revenue from auction (r, e), $R_{(r,e)}$, can be written $R_{(r,e)} = b_{(r,e)}(Z)\cdot 1_{\{Z \geq x^s\}} + e\Omega$ (recall that $Z = \max\{X_1, \ldots, X_n\}$). Using (3) and the fact that $\Omega \cdot 1_{\{Z \geq x^s\}} = \Omega$, the revenue from auction (r', e') can be written $R_{(r',e')} = R_{(r,e)} + \Psi \cdot 1_{\{Z \geq x^s\}}$, where $\Psi = (e' - e)(\Omega - 1/F_{Y_1}(Z))$. Part (i) of Corollary 1 implies $E[\Psi \cdot 1_{\{Z \geq x^s\}}] = 0$ and thus $E[\Psi | Z \geq x^s] = 0$. $R_{(r,e)}$ and Ψ are both nondecreasing functions of the signals $X_1, \ldots, X_n$ and hence are affiliated (Milgrom and Weber, 1982). The affiliation of $R_{(r,e)}$ and Ψ implies that $E[\Psi | R_{(r,e)} = q]$ is nondecreasing in q (Milgrom and Weber, 1982). Furthermore, notice that

$$E[\Psi | R_{(r,e)} > 0] = E[\Psi | Z \geq x^s] = 0,$$

since $b_{(r,e)}(x^s) > 0$ by the fact that $b_{(r,e)}(x^s) \geq v(\underline{x}, \underline{x}) > 0$.

Let Y denote the support of $R_{(r,e)}$. Define $q^* \in \underline{Y}$ such that $E[\Psi | R_{(r,e)} = q] \gtreqless 0$ as $q \gtrless q^*$; q^* is well defined, since $E[\Psi | R_{(r,e)} > 0] = 0$. Without loss of generality, normalize U so that $U'(q^*) = 1$; hence, $U'(q) \lesseqgtr 1$ as $q \gtrless q^*$. This implies

$$\begin{aligned} U(q + E[\Psi \mid R_{(r,e)} = q]) - U(q) &\le U'(q)E[\Psi \mid R_{(r,e)} = q] \\ &\le E[\Psi \mid R_{(r,e)} = q] \end{aligned} \tag{4}$$

for all $q \in \Upsilon$, with the first inequality following from concavity.

The conclusion of the theorem, $E[U(R_{(r',e')} + v_0 \cdot 1_{\{Z<x^s\}})] \le E[U(R_{(r,e)} + v_0 \cdot 1_{\{Z<x^s\}})]$, follows from the fact that $E[U(R_{(r',e')}) \mid Z \ge x^s] \le E[U(R_{(r,e)}) \mid Z \ge x^s]$. We establish this inequality as follows.

$$\begin{aligned} E[U(R_{(r',e')}) \mid Z \ge x^s] &= E[U(R_{(r,e)} + \Psi) \mid Z \ge x^s] \\ &\le E[U(R_{(r,e)} + E[\Psi \mid R_{(r,e)}]) \mid Z \ge x^s] \\ &\le E[U(R_{(r,e)}) + E[\Psi \mid R_{(r,e)}] \mid Z \ge x^s] = E[U(R_{(r,e)}) \mid Z \ge x^s]. \end{aligned}$$

The first inequality follows from Jensen's inequality for conditional expectations and the fact that conditioned on itself, $R_{(r,e)}$ is nonstochastic. The second inequality follows from (4). $E[E[\Psi \mid R_{(r,e)}] \mid Z \ge x^s] = E[\Psi \mid Z \ge x^s] = 0$ establishes the final equality. *Q.E.D.*

As is the case for Theorems 1 and 2, a generalization of Theorem 4 to the general affiliated-values case is not possible. Milgrom and Weber (1982) show that risk-neutral sellers prefer the auction with (r', e') to the auction with (r, e). Therefore, it is straightforward to see that when $X_1, \ldots, X_n$ are strictly affiliated, a seller with sufficiently slight risk aversion prefers the auction with the higher entry fee.

6. Conclusion

■ The pattern of results we have shown suggests that revenue-enhancing devices in affiliated-values auctions generally come at the expense of added revenue variability. A major concern of auction theorists has been to explain the prevalence of a few auction forms. The revenue equivalence theorem does not contribute to addressing this concern. The affiliated-values model of Milgrom and Weber (1982) predicts that risk-neutral bid takers prefer English auctions to second-price auctions to first-price auctions. Since first-price auctions are quite common, there is clearly some role for a model that predicts bid takers' preferences in reverse order to the Milgrom and Weber predictions. To some extent, this role can be played by the models of a risk-neutral bid taker facing risk-averse bidders. Our results provide another way of explaining the rather widespread choice of first-price auctions. For the case when the private information of risk-neutral bidders is independently distributed, a risk-averse bid taker prefers first-price to second-price to English auctions. In a first-price auction, he would select a lower reserve price than the expected-revenue-maximizing choice. Of course, a fuller explanation of the prevalence of a few auction forms must use this model and antecedent models of single, isolated auctions as building blocks toward models that place bid takers' choices and the behavior of potential bidders in a larger context of related transactions.

Appendix

■ Proofs of Lemmas 2 and 3 follow.

Proof of Lemma 2. Define $F_{Y_1}(\cdot \mid x)$ and $f_{Y_1}(\cdot \mid x)$ as the distribution and density functions of Y_1 conditional on $X_1 = x$. The screening level, $x^s(r)$, in a first-price auction with reserve price is implicitly defined by $\int_{\underline{x}}^{x^s} [v(x^s, y) - r] f_{Y_1}(y \mid x^s)\, dy = 0$. $x^s \in (\underline{x}, \bar{x})$, since $E[V_1 \mid X_1 = \bar{x}, Y_1 < \bar{x}] > r > v(\underline{x}, \underline{x})$. The differentiability of x^s follows from the differentiability of u and f by the implicit function theorem. Differentiating the screening level definition with respect to r, we get

$$\{[v(x^s, x^s) - r]f_{Y_1}(x^s|x^s) + \Lambda'(x^s(r))\}x^{s\prime}(r) - F_{Y_1}(x^s|x^s) = 0, \tag{A1}$$

where $\Lambda(z) = \int_{\underline{x}}^{x^s} [v(z, y) - r]f_{Y_1}(y|z)\,dy$. To see that $\Lambda'(x^s) > 0$, define $\beta(z) = E[v(z, Y_1) - r|X_1 = z, Y < x^s]$. Then $\Lambda(z) = \beta(z)F_{Y_1}(x^s|z)$, and $\Lambda'(z) = \beta'(z)F_{Y_1}(x^s|z) + \beta(z)\partial F_{Y_1}(x^s|z)/\partial z$. By definition, $\Lambda(x^s) = \beta(x^s) = 0$. Hence, $\Lambda'(x^s) = \beta'(x^s)F_{Y_1}(x^s|x^s)$. The monotonicity of v and affiliation imply $\beta'(x^s) > 0$ and, hence, $\Lambda'(x^s) > 0$. Therefore, $\Lambda'(x^s) > 0$ and $v(x^s, x^s) \geq r$ imply $x^{s\prime}(r) > 0$ for r such that $E[V_1|X_1 = \bar{x}, Y_1 < \bar{x}] > r \geq v(\underline{x}, \underline{x})$.

For $x > x^s$,

$$b(x|r) = rL(x^s(r)|x) + \int_{x^s}^{x} v(\alpha, \alpha)L(\alpha|x)\frac{f_{Y_1}(\alpha|\alpha)}{F_{Y_1}(\alpha|\alpha)}\,d\alpha,$$

where $L(\alpha|x) = \exp(-\int_\alpha^x f_{Y_1}(s|s)/F_{Y_1}(s|s)\,ds$ (see Milgrom and Weber, 1982). Using the definition of $x^s(r)$, we have

$$\frac{\partial b(x|r)}{\partial r} = L(x^s|x) - [v(x^s, x^s) - r]L(x^s|x)\frac{f_{Y_1}(x^s|x^s)}{F_{Y_1}(x^s|x^s)}x^{s\prime}(r).$$

In the case where $v(x^s, x^s) = r$, the conclusion follows easily, since $L(x^s|x) > 0$ for $x^s > \underline{x}$. If $v(x^s, x^s) > r$, then (5) implies

$$\frac{\partial b(x|r)}{\partial r} = \Lambda'(x^s)\frac{L(x^s|x)}{F_{Y_1}(x^s|x^s)}x^{s\prime}(r) > 0.$$

The inequality follows from the fact that all four terms in the expression above are positive when $x > x^s$ and $x^s \in (\underline{x}, \bar{x})$. *Q.E.D.*

Proof of Lemma 3. Assume contrary to the lemma that for some

$$x_1^* \in \operatorname{argmax}\{g_1(x)\} \quad \text{and} \quad x_2^* \in \operatorname{argmax}\{g_2(x)\},\ x_1^* \leq x_2^*.$$

It cannot be the case that $x_1^* = x_2^*$, since $0 = g_1'(x_1^*) > g_2'(x_1^*) \neq 0$. Thus, it remains to show that $x_1^* < x_2^*$ results in a contradiction. $g_1(x_2^*) - g_1(x_1^*) < 0$, since g_1 achieves a maximum at x_1^*. However,

$$g_1(x_2^*) - g_1(x_1^*) > g_2(x_2^*) - g_2(x_1^*) \geq 0,$$

since $g_1'(x) > g_2'(x)$ for all $x \in [x_1^*, x_2^*] \subset S$ and g_2 achieves a maximum at x_2^*. *Q.E.D.*

Example A1. Suppose there are two bidders such that X_1, X_2 are independently distributed according to the uniform distribution over [0, 1]. Let $v(x, y) = x + y$ and $v_0 = 0$. Notice that the assumptions of part (ii) of Theorem 3 do not hold. That is, v is not constant in its second argument. For this example, the screening level is $x^s(r) = \frac{2}{3}r$. Furthermore, the joint density $\hat{f}$ defined in the proof of Theorem 3 is $\hat{f}(z, y) = 2$ for $z > y$, $\hat{f}(z, y) = 0$ otherwise. Hence, assuming a second-price auction, in this example, for $r \in [0, \frac{3}{2}]$ we have

$$H_\Phi(r) = \int_{(2/3)r}^{1}\int_{(2/3)r}^{z} 2U_\Phi(2y)\,dy\,dz + \frac{4}{9}(3 - 2r)rU_\Phi(r),$$

and for r where the utility is differentiable, we have

$$H'_\Phi(r) = \frac{4}{9}[(3 - 4r)U_\Phi(r) + (3 - 2r)(rU'_\Phi(r) - U_\Phi(4/3r))]. \tag{A2}$$

It is easy to see that a reserve-price choice outside of $[0, \frac{3}{2}]$ will not dominate the best choice within $[0, \frac{3}{2}]$. Define $U_L(\xi) = \xi$ and

$$U_M(\xi) = \begin{cases} 2\xi, & \text{for } \xi \leq \dfrac{1}{10} \\[2ex] \dfrac{1}{10} + \xi, & \text{for } \xi \in \left(\dfrac{1}{10}, \dfrac{7}{10}\right] \\[2ex] \dfrac{8}{10}, & \text{for } \xi > \dfrac{7}{10}. \end{cases} \tag{A3}$$

Notice that U_L implies risk neutrality and that U_M implies risk aversion. The expected-revenue-maximizing reserve price in this example is $r_L^* = 3/5$, since

$$H_L'(r) = \frac{8}{27}(3 - 5r)r.$$

However, the expected-utility-maximizing reserve price when the seller has utility function U_M is $r_M^* = 7/10$. Substituting (A3) into (A2) yields

$$H_M'(r) = \begin{cases} \frac{16}{27}(3 - 5r)r, & \text{for } r \in \left[0, \frac{3}{40}\right) \\ \frac{2}{135}(-9 + 246r - 280r^2), & \text{for } r \in \left[\frac{3}{40}, \frac{1}{10}\right) \\ \frac{4}{135}(27 - 50r)r, & \text{for } r \in \left(\frac{1}{10}, \frac{21}{40}\right) \\ \frac{2}{15}(7 - 10r)(2r - 1), & \text{for } r \in \left[\frac{21}{40}, \frac{7}{10}\right) \\ -\frac{32}{45}r, & \text{for } r \in \left(\frac{7}{10}, \frac{3}{2}\right]. \end{cases}$$

The continuity of H_M follows from the continuity of U_M. (However, H_M is not differentiable at $r = 1/10$ and $r = 7/10$.) H_M achieves a maximum at $r_M^* = 7/10$, since H_M is increasing for $r \in [0, 7/10)$ and decreasing for $r \in (7/10, 3/2]$. Therefore, a risk-averse seller with utility function U_M sets a higher reserve price than a risk-neutral seller would. While U_M is not strictly concave, it is clearly possible to construct a strictly concave utility function that closely resembles U_M and has the property that a strictly risk-averse seller with that utility function will set a higher reserve price than a risk-neutral seller would.

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