An "Alternating Recognition" Model of English Auctions

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We present an alternative abstraction of an English (oral ascending) auction to the standard, in Milgrom and Weber (1982), that accords more closely with practices in some auction markets. In particular, the assumptions that exits are irrevocable and necessarily public are dropped, making endogenous the decision to compete silently and privately, or openly. In the model, the price rises in a stylization of an auctioneer alternately recognizing two bidders who affirm willingness to pay the current price. The auctioneer pays attention to other bidders only when a recognized bidder exits. Such exits may be temporary, although we construct an equilibrium in which there is no benefit to exit and reentry. The number of public exits is stochastic; frequently a losing "bidder" will remain silent, giving no indication of his willingness to pay, and hence yielding no useful inference about his private information. Thus, the source of the expected revenue advantage of English auctions over second-price auctions is only stochastically available. Moreover, when public exits are incomplete, the ordinal rank of the bidder whose private information can be inferred is unknown, making that information less valuable. Consequently, the simpler formula for expected revenue in second-price auctions may be the preferred approximation for English auctions.
(*Competitive Bidding*; *Oral Auctions*; *Auction Theory*; *Information Disclosure in Auctions*)

1. Introduction

Most auctions are oral ascending or "English" auctions (Cassady 1967 estimated 75%). The canonical model of English auctions is provided by Milgrom and Weber (1982); its primacy is attested to in influential surveys by Wilson (1993) and McAfee and McMillan (1987). Milgrom and Weber point to English auctions as superior in generating expected revenue to the other common auction forms. The revenue advantage stems from two key features of their stylization: exit from the auction is irrevocable, and continuing competition (and thus each exit) is publicly observable. Thus, all bidders still competing know the number of active bidders, and learn the price at which each losing bidder quit competing. The two highest bidders, then, effectively engage in a second-price auction after they have inferred the private information of the $n - 2$ bidders who quit earlier.[1] Making this information public is the source of the revenue advantage of Milgrom and Weber's model of English auctions (below, simply MW auctions) over second-price auctions.[2]

Neither irrevocability nor necessary publicness of exits from English auctions is a notable characteristic

[1] Introduced by Vickrey (1961), a second-price auction is a sealed-bid auction in which the highest bidder obtains the asset at a price equal to the highest rejected bid. Milgrom and Weber refer to their stylization as the "Japanese" variant of an English auction, which seems at odds with Cassady's description.

[2] This logic has been recognized, but not modeled, in Graham et al. (1990), Bikhchandani and Riley (1991), and Wilson (1993). Our initial attempt to model it was in Harstad and Rothkopf (1991).

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of the variety of auction practices that led Cassady to his 75% figure. In many oral auction markets of interest, it is possible to remain silent and yet in competition; the exit price of a silent bidder often remains his private information. In this paper, we construct and analyze an "alternating recognition" model of English auctions (below, AR auctions) which attempts to capture more closely stylized practices in many of these oral auction markets. It retains a stochastic revenue advantage over second-price auctions, but the advantage is less pronounced. For many situations, estimating revenue as if second-price rules prevailed may be a good approximation. This approximation is usually closer than that offered by MW auction predictions. Hence, this paper provides some justification for the usage of second-price auctions to model oral auction phenomena by several authors, notably Milgrom (1987).

The phrase "alternating recognition" comes from the following process, which we have witnessed frequently (in auto, furniture, real estate, bankruptcy, and estate auctions; colleagues have reported witnessing it in other contexts). An auctioneer begins an oral auction by calling out a price and asking bidders to affirm. He then recognizes an affirming bidder, and asks for a higher price to be affirmed by someone else. Having two affirming bidders, he then proceeds to raise the price in increments, alternating between his affirming pair, and paying attention to the rest of the bidders only when one of his affirming pair is no longer willing to affirm the latest price. At this point, he scans the crowd, seeking a replacement for his exiting bidder. If he finds a replacement, he continues to alternate between the newly found affirmer and the previous (still competing) affirmer. Whenever the auctioneer cannot find a replacement for an exiting bidder, the auction ends, with the asset awarded to the current affirmer at the last price affirmed.

Hence, our AR auction model has the auctioneer proceed continuously through a price interval $[\underline{p}, \overline{p}]$ chosen so that it is common knowledge that all bidders are willing to pay $\underline{p}$, and none are willing to pay $\overline{p}$.[3] At any price, the auctioneer must, in order to continue raising the price, find a pair of bidders to recognize who are willing to purchase at that price. The model assumes the auctioneer continues to use the same recognized pair of bidders as long as both are willing to affirm the price. A bidder no longer willing to be recognized "exits." The price at which he exits becomes public information, presumably because the auctioneer must "scan the crowd" for a replacement. The model assumes that the auctioneer, when searching for a replacement for an exiting bidder, selects equiprobably among those bidders willing to accept recognition. The auction ends when a replacement for an exiting bidder cannot be found.

In the equilibrium we construct, bidders choose not to avail themselves of the wider variety of stratagems allowed by the model's richness. If still competing, they accept recognition rather than remain silent. They choose not to reenter after exiting. Section 6 provides the distribution of the number, and ranks, of exit prices revealed during the auction, and gives an example in which the revenue advantage of English auctions relative to second-price auctions is much diminished. This occurs because the revelation of exit prices is typically incomplete, and is substantially further diminished because ranks of those bidders whose exit prices are revealed typically remain undisclosed. Thus, Milgrom and Weber's analysis is seen to depend both on revelation of losing bidders' private information, and on revelation of the ordinal rank of that private information, a distinction that is revealed by analyzing a model that allows bidders to choose to remain silent while still competing.

2. The Auction Environment

The environment for our model is that introduced in Milgrom and Weber (1982) as the "General Symmetric Model." A discussion of the model, its assumptions and relation to the literature, is presented in pages 1090–1100 of their paper; only key provisions are mentioned in this section.

[3] Note that we use the term "bidders" to refer to the set of players, those who would be bidding at the start of an MW auction, as is typical in the auction literature—even though an observer may not see such a "bidder" ever bid.

An indivisible asset is sold by auction. Each bidder $i \in \mathbb{N} = \{1, \ldots, n\}$ possesses private information represented by a real-valued *signal* X_i, with $X = \{X_1, \ldots, X_n\}$. In addition, $S = \{S_1, \ldots, S_M\}$ represent relevant but unobserved random variables. The asset's value to bidder i, V_i, may depend upon variables unknown to him.

Assumption 1. *The joint distribution of the random variables* (S, X) *is continuous, strictly affiliated, and exchangeable in the* X_i*s.*

Assumption 2. *There are nondecreasing, continuous value functions* U, U_i, $i \in \mathbb{N}$, *so that* $V_i = U_i(X, S) = U(X_i, \{X_j\}_{j \neq i}, S)$, *where* U *is symmetric in the* X_j*s,* $j \neq i$. *That is, each bidder's value is a symmetric function of rivals' information, and depends on nonparticipant appraisals in the same way.*

Roughly, affiliation implies that higher realizations for some subset of the variables (x, s) makes higher realizations of any other subset more likely.[4]

The seller and all bidders are assumed risk-neutral. Using symmetry, we focus on Bidder 1, and let $Y := \max\{X_2, \ldots, X_n\}$. For simplicity, we assume throughout that the event $\{X_1 = Y\}$ has zero probability. As do Milgrom and Weber (1982), we ignore the discreteness of bidding intervals (cf. Rothkopf and Harstad 1994b).

3. The Basic Argument

The equilibrium in the MW auction has each bidder exit when the price reaches a conditional expectation of asset value, where the conditioning inverts the exit prices of earlier exiting bidders to infer their signals X_j. In the AR auction, no comparable inference is available about signals of bidders who remain silent. A silent bidder may still be in competition, or may have, in effect, exited silently before the first exit, between any pair of exiting bidders, or after the last exit but below the current price.

The symmetric equilibrium in a second-price auction, without any revelatory dynamics, has a bidder make a specific inference about the highest rival signal (that it is equal to his) for the sole purpose of producing the price at which this bidder is indifferent between winning and losing. The only inferences made about other rivals are definitional: they cannot have higher signals than the rival with the highest rival signal. This is correspondingly the only inference that a bidder in an AR auction (in symmetric equilibrium) makes about silent bidders: that eventually the other recognized bidder will be the rival observing the highest rival signal.

The essential intuition of Milgrom and Weber's revenue comparisons runs as follows. Suppose a seller in a second-price auction were to publicly reveal information that is affiliated with asset value, such as the lowest of the n signals. Each bidder could have calculated before the announcement what he expected the public information announcement to be. Suppose, parallel to the calculations that went into his bidding, each bidder calculated the expected announcement under the assumption that his signal was highest. Then, on average, the bidder who observed the highest signal would not be surprised. However, his leading rival, the price-setter, on average views the announcement as "good news": a signal that is ordinally $n - 2$ ranks below his rank will on average be higher than his calculation of the expectation of a signal ordinally $n - 1$ ranks below his *assumed* first rank. So on average the price-setter views asset value as higher and bids more aggressively. This explains the phenomenon that Milgrom and Weber (1982) call the Linkage Principle. This intuition gets applied $n - 2$ times in the MW auction, as the last two bidders observe all their rivals exiting. It gets applied only upon exits in the AR auction.

For example, suppose $n = 6$ and the auction continues after two recognized bidders have exited. Then their exit prices yield inferences about two signals, but it is not clear what the ranks of these two signals are: the first bidder to exit could have observed the fourth-, fifth-, or sixth-highest signal (the third-highest signal is only ruled out by the second exit). Indeed, the auction will finish without rank-order information about these signals, unless it continues

[4] Cf. the discussion in Milgrom and Weber (1982, pp. 1098–1100, their Appendix) and Riley (1988). Strict affiliation is technically a stronger assumption than needed for our results, simplifying notation and proofs.

past the $(n - 2)$nd exit. Without rank-order information, information that rivals observed particular signals is much less valuable and less likely to be interpreted as good news by the price-setter. We provide an example in §6 illustrating the likely result: that expected revenue in an AR auction may not be much higher than in a second-price auction.

The principal concern in characterizing equilibrium bidding in an AR auction is whether to remain silent when that is an option. Cassady's discussion of bidders' practices in English auctions notes:

> Another issue related to bidding strategy is whether to be bold or cautious in initial bidding. ⋯ [Sometimes] a prospective buyer, even though determined to purchase an item, bids tentatively and cautiously in order to feel out the opposition. He hopes that by indicating a low regard for the offering he will lull opponents into a false sense of security. (1967, p. 147)

The cautious bidder, however, must eventually indicate sufficient regard for the item to outbid his rivals. The proof of Theorem 1 below shows that, in a rational, symmetric environment, a bidder does not gain by caution, if caution is interpreted as remaining silent when rivals are willing to become the recognized bidders.[5] The intuition is that to win, a bidder will eventually have to be recognized. If he foregoes an opportunity to be recognized, he will increase the expected number of rivals whose exit prices become publicly known, and thus the expected price he will pay if he wins.

A caveat is in order as the analysis here rests more heavily than elsewhere on the presumption that the current auction can be analyzed in isolation. To wit, we assign zero profit to all events where a bidder fails to win. In some contexts, however, bidders would prefer remaining silent and losing to exiting publicly and losing, if the latter were to reveal private information. They may face the same rivals again in later auctions. If a bidder's valuations are correlated across auctions (e.g., due to valuation methods or inventories), he will care per se if rivals learn his estimate of value for this asset. Thus, if a rival could infer some useful information about a used-car dealer's inventory from his bidding, that dealer will prefer to avoid giving away the information in an auction he loses—indeed, his expected profit when he wins would have to be high enough to cover the expected disadvantage of rivals learning such information.

The next section develops the symmetric equilibrium of the AR auction under the assumption that exits are irrevocable. Some intricate definitions are required, but the logic follows exactly the argument just outlined. The following section will show that irrevocable exits were assumed merely for expositional convenience, in that a bidder will have no reason to reenter after exiting, and cannot gain by a temporary exit and re-entry if rival bidders ignore the exit once it is seen to be temporary.

4. Equilibrium Bidding

Even temporarily ignoring the possibility of revocable exits, the option to compete silently adds informational richness, and the formal analysis of the equilibrium is more cumbersome. However, all this section does is show that accepting recognition when offered, below the price that would be bid in a second-price auction with corresponding informational inferences, constitutes equilibrium behavior. Define:

$$v(x, y) = E[V_1 | X_1 = x, Y = y]. \tag{1}$$

This is expected asset value conditional on Bidder 1 observing x and on y being the highest signal observed by a rival. Let $b_0(x) = v(x, x)$. Then $(b_0, \ldots, b_0)$ is the unique symmetric equilibrium of the second-price auction (Matthews 1977, Levin and Harstad 1986).

We suppose Bidder 1 observes $X_1 = x$, and begin defining the equilibrium strategy for the AR auction with this rule:

$\mathbb{R}0$. If $b_0(x) \geq \underline{p}$, and if recognized by the auctioneer, accept recognition. Continue to accept recognition so long as the other bidder does not exit, and so long as $p \leq b_0(x)$. Exit at $p = b_0(x)$.

[5] Cassady's "caution" could also be interpreted as *pedestrian* bidding, i.e., being no more aggressive than necessary to continue competing. Pedestrian bidding can be equilibrium behavior in the following sense. Suppose the options available to a recognized bidder were expanded to include skipping over allowable bid levels. Rothkopf and Harstad (1994b) consider this option in their model of discrete bidding increments, and find that pedestrian bidding is an equilibrium when the increments are not too large.

Let $\theta_1(p_1)$ be the event $\{\exists j \neq 1 | p_1 = b_0(X_j)\}$, which is the inference that a bidder can draw if a rival follows $\mathbb{R}0$, is recognized and exits at p_1. Define

$$b_1(x, p_1) = E[V_1 | X_1 = Y = x,\ \theta_1(p_1)],$$

and continue defining the equilibrium strategy with this rule:

$\mathbb{R}1$. Suppose the first recognized bidder to exit does so at price p_1. If $b_1(x, p_1) \geq p_1$, continue to accept recognition if already recognized, and accept recognition if now recognized by the auctioneer. Continue to accept recognition so long as the other bidder does not exit, and so long as $p \leq b_1(x, p_1)$. Exit at $p = b_1(x, p_1)$.

Exit price definitions are recursive:

$$\theta_2(p_1, p_2, k_1) = \{\exists j \in \mathcal{N} \backslash \{1, k_1\} | p_2 = b_1(X_j, p_1)\},$$

$$\theta_r(p_1, \ldots, p_r, k_1, \ldots, k_{r-1}) = \{\exists j \in \mathcal{N} \backslash \{1, k_1, \ldots, k_{r-1}\} | p_r = b_{r-1}(X_j, p_1, \ldots, p_{r-1})\},$$

$$b_r(x, p_1, \ldots, p_r) = E[V_1 | X_1 = Y = x, \theta_1(p_1), \ldots, \theta_r(p_1, \ldots, p_r, k_1, \ldots, k_{r-1})]. \qquad (2)$$

In these definitions, $\theta_2(p_1, p_2, k_1)$ is the event that some bidder other than Bidder 1 or Bidder k_1 observed a signal that caused him to exit at price p_2, once he had drawn the appropriate inference from a previous exit at price p_1. The value of k_1 is irrelevant, as this is only a placeholder, to make sure that the bidders exiting at prices p_1 and p_2 are distinct bidders. Similarly, θ_r is the event that some bidder other than Bidder 1 or a previously exiting bidder observed a signal that caused him to exit at price p_r, given that he saw previous bidders exit at prices $p_1, \ldots, p_{r-1}$. The resulting exit functions b_r are then conditional expected-value calculations depending upon inferences from exit prices, and assuming that the highest signal observed by a rival who has not yet exited (the other recognized bidder, or perhaps a bidder who has been silent so far) equals the signal that this bidder himself observed. Notice that all these definitions only make sense if $p_1, \ldots, p_r$ is a nondecreasing sequence of prices. The affiliation assumption implies that all b_r functions, $r = 0, \ldots, n-2$, are increasing and thus invertible, and also implies a useful sort of consistency:

$$\hat{p} = b_1(x, p_1) \text{ f } \lim_{p_2 1 \hat{p}} b_2(x, p_1, p_2) \geq \hat{p}. \qquad (3)$$

In words, a bidder who was about to exit does not lower his planned exit price when a rival exits just ahead of him. The consistency condition corresponding to Equation (3) holds for every stage r, but for simplicity is stated only for $r = 2$.

Continuing, for $r = 2, \ldots, n-2$, define the equilibrium strategy with the rules:

$\mathbb{R}r$. Suppose recognized bidders have exited at prices $p_1, \ldots, p_r$. If $b_r(x, p_1, \ldots, p_r) \geq p_r$, continue to accept recognition if already recognized, and accept recognition if now recognized by the auctioneer. Continue to accept recognition so long as the other bidder does not exit, and so long as $p \leq b_r(x, p_1, \ldots, p_r)$. Exit at $p = b_r(x, p_1, \ldots, p_r)$.

Let $\mathbb{R}$ denote the set of rules $\{\mathbb{R}0, \mathbb{R}1, \ldots, \mathbb{R}n-2\}$.

Theorem 1. *Given Assumptions* 1 *and* 2, *the strategy profile* $(\mathbb{R}, \ldots, \mathbb{R})$ *is a symmetric equilibrium point of the Alternating Recognition auction with irrevocable exit.*[6]

Proof. Let bidders $2, \ldots, n$ follow $\mathbb{R}$, and consider Bidder 1. With irrevocable exit, Bidder 1's alternatives to $\mathbb{R}$ involve exiting earlier, or continuing to accept recognition longer, at some stage r. First suppose that bidder 1 considers some alternative strategy $\hat{\mathbb{R}}$, which exits earlier at some stage ρ, with planned exit price $\hat{p} < b_\rho(x, p_1, \ldots, p_\rho)$. Suppose the event ψ: {the auction reaches price $\hat{p}$ when ρ bidders have exited}. Given ψ, strategy $\hat{\mathbb{R}}$ exits to attain 0 profit.

Under $\mathbb{R}$, Bidder 1 wins only in the event $\Psi := \{\psi\} \cap \{X_1 > Y\}$. Note $\Pr[\Psi] > 0$. Given Ψ, at some stage $r \geq \rho$, the auctioneer recognizes the two highest-signal holders. When this rival exits at price $b_r(y, p_1, \ldots, p_r)$, bidder 1 wins, attaining expected profit:

[6] There are other equilibria. This equilibrium is unique in the class of symmetric equilibria; the proof of this claim is laborious and unenlightening, and so is omitted. The general uncertainty over whether the two recognized bidders are the last two competitors, though, is both a basis for a unique symmetric equilibrium and for a diminished scope for asymmetric equilibria.

$$E[V_1|X_1 = x,\ Y = y,\ \theta_1(p_1), \ldots, \theta_r(p_1, \ldots, p_r, k_1, \ldots, k_{r-1})] - E[V_1|X_1 = y,\ Y = y,\ \theta_1(p_1), \ldots, \theta_r(p_1, \ldots, p_r, k_1, \ldots, k_{r-1})]. \quad (4)$$

By affiliation, the first term (expected asset value) exceeds the second (price), since $x > y$. Hence, Bidder 1 loses by exiting earlier than $\mathbb{R}$ instructs.

By a parallel argument, in the event that Bidder 1 competes longer than $\mathbb{R}$ calls for, and wins, his expected profitability is again given by Equation (4) above. In this case $x < y$, and the gain due to deviating from $\mathbb{R}$ is again negative. Thus, neither exiting sooner, nor continuing to accept recognition longer than $\mathbb{R}$ instructs, can increase bidder 1's expected profit. □

5. Revocable Exits

We now introduce the realistic feature of revocable exits: a bidder who decides not to accept recognition at some price nonetheless has the option of reentering the fray via accepting recognition later at a higher price. Two possible reasons for reentering are: the additional information revealed by exits since this bidder exited may lead to a favorable asset revaluation, and the bidder may hope to gain by using an early exit to convince rivals that some bidder thinks asset value is quite low.

Nonetheless, with an appropriate specification of response to reentry appended to $\mathbb{R}$, we can derive a symmetric equilibrium in which the opportunity to exit and later reenter is never invoked. Equilibrium behavior thus follows exactly the same straightforward, "nonstrategic" path as before.

Suppose that bidders have exited at prices $p_1, \ldots, p_r$, but that one of these now reenters: Bidder k who exited at price p_k again becomes one of the two bidders accepting recognition. All bidders respond by ignoring the exit at price p_k. That is, they revert to rule $\mathbb{R}r - 1$, with $r - 1$ exit prices $p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_r$.[7] The next time bidder k exits is (tentatively) treated as indicative of his private information, as if it were the only time he exited.

To see that the behavior described constitutes a symmetric equilibrium, it suffices to show that Bidder 1 cannot attain a higher expected profit by any timing of exiting and reentering than by following rule $\mathbb{R}$. To prove this, initially note that the specified reaction to reentry renders irrelevant the timing of the original exit and the reentry: all rival bidders behave as if the exit had never occurred. Clearly, Bidder 1 cannot gain by reentering only to exit earlier than he would have if following strategy $\mathbb{R}$. If he exits precisely as instructed by $\mathbb{R}$ (following his reentry), by construction his expected profitability is no higher.[8] Thus, exit and reentry cannot undermine the purported equilibrium unless Bidder 1 can gain by competing longer after reentry than instructed by $\mathbb{R}$. Since his last rival's behavior is unaffected by the reentry, and bidding longer only yields incremental expected profit by winning when $X_1 < Y$, the relevant calculation is still (4) above, and it is negative. Thus, competing longer after reentry is unprofitable, and any single bidder has no incentive to exit earlier than $\mathbb{R}$ specifies, or to reenter after he exits. If reentries are responded to in this way, then the model's predictions are unaffected by introducing a revocable exit option.

We have no rigorous proofs to offer, but have considered two possibilities. First, suppose bidders have difficulty tracking an exiting and reentering bidder. We believe that the lack of incentive for exit and reentry is probably a robust phenomenon, because increasing the number of apparent final exits

[7] However, they take account of the fact that bidders exiting at prices $p_{k+1}, \ldots, p_m$ (the "later exiting bidders") believed there was an exit at p_k; in other words, the inference about the signals of these later exiting bidders that had been made before the reentry is not altered. If any of the later exiting bidders is now willing to pay the current price, they may revert to willingness to accept recognition. They need not, as they have no remaining private information and no profit opportunities given that the other currently recognized bidder is following $\mathbb{R}$. Indeed, all bidders can infer that this other currently recognized bidder observed a higher signal than any of the later exiting bidders.

[8] His expected profitability is in fact lower in the event that his temporary absence from the recognized pair of bidders has led to the revelation of some exit prices by rivals who otherwise would have been silent.

Table 1 Probabilities of r Prices Being Revealed

(1) n	(2) avg	(3) frac	(4) $n - 2$	(5) 0	1	2	3	4	5	6
2	0			1	0					
3	0.67	0.67	0.666667	0.3333	0.6667	0				
4	1.17	0.58	0.333333	0.1667	0.5000	0.3333	0			
5	1.57	0.52	0.133333	0.1000	0.3667	0.4000	0.1333	0		
6	1.90	0.48	0.044444	0.0667	0.2778	0.3889	0.2222	0.0444	0	
7	2.19	0.44	0.012698	0.0476	0.2175	0.3571	0.2698	0.0952	0.0127	0
8	2.44	0.41	0.003175	0.0357	0.1750	0.3222	0.2917	0.1389	0.0333	0.0032
9	2.66	0.38	0.000705	0.0278	0.1440	0.2895	0.2985	0.1728	0.0568	0.0099
10	2.86	0.36	0.000141	0.0222	0.1208	0.2604	0.2967	0.1980	0.0800	0.0193
12	3.21	0.32	0.000004	0.0152	0.0888	0.2130	0.2809	0.2283	0.1205	0.0422
15	3.64	0.28	0.000000	0.0095	0.0606	0.1627	0.2485	0.2437	0.1633	0.0774
20	4.20	0.23	0.000000	0.0053	0.0368	0.1118	0.1994	0.2361	0.1987	0.1238
50	6.00	0.12	0.000000	0.0008	0.0073	0.0298	0.0755	0.1338	0.1780	0.1860

observed by rivals would stimulate them to increase their valuations and hence their planned exit prices. Second, suppose an auctioneer attempts to alternate among three or more bidders. We observe that such behavior is not common in practice and that bidders would probably have no incentive to seek or accept recognition as a third or fourth bidder, because the public exit of at least one of the currently-recognized bidders is unavoidable.

6. Information in Exit Prices

The number of exit prices revealed is stochastic. The only way a potential bidder's exit price is revealed is if he becomes a recognized bidder and exits before the auction is over. To clear up terminology, we will refer to an exit price as being *revealed* if it becomes publicly known early enough in the auction that remaining bidders have an opportunity to incorporate this information in their bidding. By definition, the second-highest bidder's exit price is never revealed, and the number of prices revealed, r, will be a random variable with support $\{0, \ldots, n - 2\}$.

This section presents and discusses numerical illustrations of the number of exit prices revealed, and their influence with incomplete ordinal ranks. A derivation and formulas are contained in the Appendix.

Table 1 provides the numerical results. Column 1 specifies the number of bidders, n. Column 2 gives the expected number, $\bar{r}$, of prices revealed, and Column 3 gives $\bar{r}/(n - 2)$, the fraction of revealable prices actually revealed. The remaining columns are probabilities. Column 4 gives the probability that all $n - 2$ possible prices are revealed. The next seven columns give the probabilities that exactly $r = 0, 1, \ldots, 6$ prices are revealed. These results do not, of course, depend upon the form of the distribution of private values (except for the assumption that it is nonatomic).

The entries in Table 1 can be calculated by assuming, without loss of generality, that the auctioneer begins by choosing a random order in which he will call upon the potential bidders to ask them to accept recognition. From the viewpoint of an omniscient observer, the bidders can be labelled by the rank order of their private signals: Bidder 1 has the highest signal, Bidder 2 the second-highest, and so on. If there are n bidders, then there are $n!$ equally likely permutations of these labels corresponding to random orders in which the bidders might be called on by the auctioneer, and for each such permutation, the number of revealed exits is determinate. For example, suppose $n = 4$ and the random order is 2–1–4–3. Then the first two bidders recognized are those who happen to have the two highest signals, and no exits will be revealed. Whereas, if the order is 3–1–2–4, Bidders 3 and 1 will be called on first, Bidder 3 will eventually exit, Bidder 2 will be called on next, and there will be no further exits revealed. Note that the order 3–1–4–2 yields the same outcome, as Bidder 4 declines recognition when Bidder 3 exits. For the case $n = 4$, there are 24 such permutations, of which 4 lead to no revealed exits, 12 to one revealed exit, and 8 to two revealed exits, yielding the probabilities 0.1667, 0.5, and 0.3333 in the third row of the table.

A glance down the third column of Table 1 shows that the fraction of revealable exit prices revealed falls steadily as the number of potential bidders increases. (And when n is 50, this fraction falls to 0.12.) A glance down Column 4 shows that the probability that all of the exit prices are revealed rapidly becomes negligible. A slightly more subtle observation is that the modal (most likely) number of prices revealed closely approximates the mean number shown in Column 2.

An expected revenue calculation for an AR auction is difficult even in simple examples. Clearly, expected revenue reaches the level of the MW auction only when all $n - 2$ exit prices are revealed. The public information inferred from at least one revealed price raises expected revenue above the level of the second-price auction. This section presents an example illustrating that the revenue enhancement associated with learning a proper subset of the $n - 2$ exit prices is likely to be seriously weakened by the lack of certainty over the ordinal rank of the bidders exiting.

For concreteness, suppose there are four bidders, and that signals are unbiased. In equilibrium, each bidder initially recognized at the beginning of the auction plans to exit at the bid he would have tendered in a second-price auction. Suppose one bidder exits at price p, another bidder is recognized, and the auction continues. At this stage, it is common knowledge that one of two events have occurred, ξ_3: {the third-highest bidder has exited}, or ξ_4: {the fourth-highest bidder has exited}. A bidder still competing can adopt a conditional bidding strategy that depends on whether a second exit is revealed. Conditional on a second exit being revealed, the probability of ξ_3 is zero, and the competing bidder's exit price calculation is simplified by the knowledge that the two observed exit prices are those of the two bidders with the lowest signals. Conditional on no further exit being revealed, the competing bidder's exit price calculation must take into account a nonzero probability for ξ_3. With no further information, the conditional probability of ξ_3 given no further exit would be $\frac{2}{3}$. (Recall that in 12 of the 24 permutations there will be exactly one exit, and note that in 8 of these 12, the exiting bidder is the one with the third-highest signal.)

However, each bidder is planning his exit price under the assumption that his signal X_i is tied for highest, and he rationally incorporates this assumption, and the implications of how far the first exit price is below this assumed tied-for-highest signal, into his estimate of the probability that the bidder observed exiting was the third-highest bidder: $\rho_3(x, p) := \Pr[\xi_3 | X_1 = Y = x, \theta_1(p)]$; recall that $\theta_1(p)$ is the information that can be inferred from a first exit at price p. These implications lead him to attach a higher probability to the event ξ_3 when p is nearer to x.

The revenue enhancement relative to the second-price auction stems from the second-highest bidder responding to a higher p by raising his planned exit price, the effect Milgrom (1981) calls "good news." However, to the extent that a higher p is merely translated into a higher probability of ξ_3, the response to this news is weaker.

To illustrate, consider a common-value auction in which the common value V is distributed uniformly on [1, 999], and each bidder's signal X_i is i.i.d. uniformly on $[v - 1, v + 1]$, given $V = v$. Expected revenue is simply 500 less the winner's expected profit, so we focus on expected profit.[9] Assuming uniform distributions eases calculations, but carries the pathology that a bidder who learned the lowest rival signal rationally ignores higher rival signals.

Table 2 reports on this example. When no information about the 3rd- and 4th-highest signals is revealed, as in a second-price auction (or when the two highest bidders happen to be recognized first), expected profit is 0.3. If the price-setting bidder can infer the lowest bidder's signal, as in the MW auction, this undercuts the private information of the winner, reducing expected profit to 0.2, shown in the second row of Table 2. In §3, we explained that the price-setter increases his

[9] All expected profit calculations harmlessly approximate by ignoring complications arising when the second-highest signal $Z_2 \in [1, 2] \cup [998, 999]$.

Table 2 **Impact of Ordinal Rank Uncertainty**

Situation	Signal Revealed	Ordinal Rank	Expected Profit	Responsive to Good News
MW English	3rd & 4th	Known	0.2	0.5
Hypothetical	4th	Known	0.2	0.5
Hypothetical	3rd or 4th	Known	0.2444	0.389
AR	Stochastic	Unknown	0.2596	–
Hypothetical	3rd	Known	0.2667	0.333
Hypothetical	3rd or 4th	Unknown	0.2858	0.24–0.35
2nd Price	None	–	0.3	–

Note. 4-bidder example with uniform distributions. Expected profit depends on which exit prices are revealed, but also on what is known about the rank order of exiter's signal.

exit price as the information inferred from earlier exit prices is more positive. This feature is shown in the right-hand column of Table 2, called responsiveness to "good news." What this column reports is the partial derivative of the equilibrium bid function with respect to the exit price (which is $\frac{1}{2}$ in the MW auction, or whenever the first exit is known to be by the bidder with the lowest signal). The more the price-setter responds to a higher exit price in his own bidding aggressiveness, the more the winner's profit is undercut.

To clarify the role of information about the ordinal rank of revealed exit prices, we momentarily make a hypothetical move outside the AR model. First, suppose one bidder exited, and it was common knowledge to the pair of bidders then recognized that the remaining bidder was still interested at the exit price. In this event (ξ_4), the exiter will be inferred to have been the lowest bidder, and expected profit with no further revealed prices will be the 0.2 corresponding to the MW auction. Second, suppose one bidder exited, and it was common knowledge to the pair of bidders then recognized that the remaining bidder was not interested at the exit price. In this event (ξ_3), the exiter will be inferred to have been the third-highest bidder, and expected profit with no further revealed prices will be the 0.2667 shown in the sixth row of Table 2. The bid function used will have a partial derivative with respect to the exit price of $\frac{1}{3}$.

Third, suppose one bidder exited, and it was common knowledge to the pair of bidders then recognized whether or not the remaining bidder was still interested at the exit price. This event is referred to in Table 2 as "3rd or 4th" signal revealed, ordinal rank "known" (fourth row). From some behavior outside the model, the two recognized bidders have both inferred the rank of the exiting bidder. Given the probabilities of that rank inferred from the lack of a second exit price, on average profit in this event is 0.2444, and the weighted average responsiveness to good news is 0.389.

These hypothetical events do not happen in an AR auction. With four bidders and a single exit price, the ordinal rank of the exiting bidder is unknown. Bidders rationally infer this rank according both to ex ante probabilities and to how close to their planned exit price the exit occurred. While a higher exit price associated with a known ordinal rank would be good news about asset value, and the bid function would be quite responsive, this impact is undermined. A higher exit price is viewed as more likely to represent the third-highest bidder, and thus as less indicative of good news. This is reported as "3rd or 4th" signal revealed, ordinal rank "unknown" (seventh row). Expected profit is higher, 0.2858, and responsiveness (which is nonlinear), ranges from around 0.24 to 0.3 over the relevant range of exit prices.

In this example,

$$74.6\% = \frac{100(0.2858 - 0.2444)}{(0.3 - 0.2444)}$$

of the impact of the information inferable from one exit price is lost due to unknown rank of the revealed signal.

Taking account as well of the events in which no exit prices or two exit prices are revealed, expected profit for an AR auction in this example is 0.2596. This represents, overall, a loss of 60% of the impact that exit-price information has in the MW auction, relative to the second-price auction. We suspect that the features of this example that faciliate computation (few bidders and uniform distributions) enhance the revenue advantage of an AR auction relative to a second-price auction.

7. Comments on Extensions and Concluding Remarks

In practice, exits from English auctions in many markets are revocable and not necessarily public. With these realistic features introduced in AR auctions, the engine driving the expected revenue advantage of MW auctions over second-price auctions is only stochastically available, usually without critical ordinal information about the ranks of the private information inferred from public exits. We have eschewed unnecessary complications in making this point. Nonetheless, the sort of extensions considered in Milgrom and Weber (1982) and McAfee and McMillan (1987) can readily be handled in this model. In particular, reserve prices and entry fees can be treated exactly as in the Milgrom-Weber analysis, and royalties as in Riley (1988). Symmetric bidder risk aversion could also be introduced in parallel fashion to the treatment in Milgrom and Weber (1982). AR auctions can be considered in the model of bid-taker risk aversion proposed by Waehrer et al. (1998); it is a straightforward extension to show that a risk-averse bid-taker would prefer a first-price auction to an AR auction when the private information of risk-neutral bidders is independently distributed.

The auction markets that motivate this model principally sell several assets sequentially in a single session. In principle, a model of sequential AR auctions would not create new problems beyond a model of sequential MW auctions. The added stochastic elements would encumber computations. Another feature of such markets is that the number of bidders is typically not a parameter, but a variable determined by endogenous response to the expected profitability of participating. AR auctions could readily be worked into the endogenous bidder participation models of Harstad (1990, 1993): as they are more extractive mechanisms than second-price but less extractive than MW auctions, they would attract on average fewer bidders than second-price, but a greater number than MW auctions. When the costs of gathering information about asset value are low and potential bidders numerous, this would extend the expected revenue comparisons from the fixed-number-of-bidders case. Conversely, when few potential bidders faced large bid preparation costs, a seller who had a choice between an AR auction and some device which actually implemented an MW auction would prefer the AR auction, as the impact on revenue of enhancing the number of participants would outweigh the impact of a less extractive mechanism.

Milgrom and Weber (1982) show that their idealization of an English auction is preferred to any other standard auction form by a seller seeking to maximize expected revenue from a given number of risk-neutral bidders. Their demonstration has been widely cited and has lent some understandable primacy to English auctions in the literature. The model they chose, however, diverges from practices we've observed in many markets, implying more information is available to the final two bidders than seems an appropriate stylization of these markets. We do not regard the alternative offered here as more than an idealization, but we think it bears attention for two reasons. First, it more directly resembles many markets transacting via oral ascending auctions.[10] Second, suppose the definition of an English auction model incorporates a requirement of some relevant dynamics, in that bidders update their planned exit prices during the auction.[11] Then this model and the MW model might reasonably be said to span the spectrum: the MW model is as informationally rich as an English auction could be, and the AR model is about as informationally sparse as an English auction can be and still exhibit exit-price updating.

[10] In the AR model, being recognized is an act observed by other bidders; being willing to be recognized is not. This accords with many common practices. For example, in auto auctions we have witnessed, it is common for used-car dealers to stand facing the auctioneer with arms folded, and then to indicate willingness to affirm by a raised thumb. This gesture is clearly visible from directly in front of the bidder, but not from behind or either side. The tendency of professionals to remain poker-faced while bidding is much more general than this specific practice.

[11] This supposition rules out a model in which bidders had the option to communicate their willingness-to-pay to the auctioneer in complete privacy. Such a model would exhibit two quite different symmetric equilibria, one in which bidders exited at the prices they would have bid in a second-price auction, and another in which bidders did not exercise this option, and followed the equilibrium behavior of an MW auction (Harstad and Rothkopf 1991).

Bidders in alternating recognition auctions cannot remain silent and win, but some bidders typically remain silent while still competing to the highest price at which they can avoid the winner's curse; their exits, and with them their information, remain private. The consequently diminished information revelation in AR auctions relative to MW auctions reduces expected revenue. Whenever there is at least one public exit and continued competition, a slight (but probably only slight) expected revenue advantage over a second-price auction with the same number of bidders is attained.

It strikes us as unhealthy to expect one model to be the appropriate abstraction of the wide variety of transacting procedures called English auctions (cf. Rothkopf and Harstad 1994a). There may be cases in which the procedures are so informationally rich and bidders' opportunities to communicate privately their interest in the asset are so limited that the Milgrom and Weber model may be a close approximation. In many others, the limited information flow may produce incentives more like the AR auction.

In many cases, bidders may obtain some information from other aspects of bidding behavior not modeled here. To give one example, used-car dealers at auto repossession auctions sometimes silently appear to watch earnestly the bidding on one vehicle until some point, and then seem to turn their attention to the next vehicle to be sold. This behavior is not necessarily uninformative, but we would be reluctant to attach as sharp a meaning to the precise price at which one turned to examine the next car as if he had been one of the alternating recognized bidders who in fact refused to be recognized at that price. We are not arguing for modeling these auctions in sufficient detail to incorporate all such mannerisms, but merely for caution in applying the model to situations which are in fact informationally richer.[12]

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Appendix

This appendix develops the formulas behind the numbers in Table 1.

Denote by $\mathcal{P}_r^h$ the probability that r of the h highest prices are revealed. (Note that the presence of bidders with lower valuations has no effect on the number of higher prices revealed. Note also that h can be set equal to the number of bidders n.) This probability can be determined by a simple iterative process. We know that $\mathcal{P}_0^2 = 1$ and $\mathcal{P}_{h-1}^h = 0$ for $h \geq 2$. For $h \geq 3$, to determine the $\mathcal{P}_r^h$s from the $\mathcal{P}_r^{h-1}$s, recall that the hth bidder added has a lower signal than all $h - 1$ prior bidders. Thus, this hth bidder's exit price will be revealed if and only if he is one of the first two bidders recognized, which happens with probability $2/h$. Using this, $\mathcal{P}_r^h$ can be calculated iteratively:

$$\mathcal{P}_0^h = \mathcal{P}_0^{h-1}(h-2)/h, \quad \mathcal{P}_r^h = \mathcal{P}_r^{h-1}(h-2)/h + 2\mathcal{P}_{r-1}^{h-1}/h,$$

$$r = 1, \ldots, h-2.$$

Thus, letting $\mathcal{P}^h$ denote the vector $(\mathcal{P}_0^h, \ldots, \mathcal{P}_{h-2}^h)$, calculations yield $\mathcal{P}^3 = (1/3, 2/3)$, $\mathcal{P}^4 = (1/6, 1/2, 1/3)$, $\mathcal{P}^5 = (1/10, 11/30, 2/5, 2/15)$, and $\mathcal{P}^6 = (1/15, 5/18, 7/18, 2/9, 2/45)$. Passing through the iteration, the probabilities for all, and for none, of the $n - 2$ lowest bidders' exit prices to be revealed are:

$$\mathcal{P}_{n-2}^n = \frac{2^{n-1}}{n!} \quad \text{and} \quad \mathcal{P}_0^n = \frac{2}{n(n-1)}.$$

Thus, these are the probabilities of attaining expected revenue matching the MW prediction, and matching a second-price auction. For $n > 3$, the bulk of the probability goes to strictly intermediate outcomes, where exit prices are revealed for a nonempty, proper subset of the $n - 2$ bidders who neither win nor determine the price.

The general formula for $\mathcal{P}_r^h$, $r > 0$, implied by the recursion is

$$\mathcal{P}_r^h = \frac{2^{r+1}}{h(h-1)} \sum_{i_1=3}^{h+1-r} \sum_{i_2=i_1+1}^{h+2-r} \cdots \sum_{i_r=i_{r-1}+1}^{h} \left\{ \prod_{j=1}^{r} (i_j - 2)^{-1} \right\}. \tag{5}$$

Let $\bar{r}_h$ denote the mean value of r. The formula for $\bar{r}_h$ is harmonic: $\bar{r}_h = -3 + 2 \sum_{i=1}^{h} i^{-1}$. As noted, lower-ranked signals are less likely to be revealed in exit prices; when there are n bidders, the average rank-order of a signal revealed is $2(n-2)/\bar{r}_n$.

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